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# THE ROBUSTNESS OF TWO-GRID COUPLED CELLULAR NEURAL NETWORKS TO PARAMETERS VARIATION

BY

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**Abstract.** The paper presents several results concerning the stability of double grid coupled Cellular Neural Networks (CNN's) linearized in the central linear part of the cell characteristic for variable parameters using Gershgorin's theorem. It is shown that the stability margins towards the right-hand side of complex plane are larger than expected according to simulations, the cell parameter variations within certain limits will preserve the instability of the array.

**Key words:** Cellular Neural Networks; eigenvalues; Gershgorin circles.

## 1. Introduction

Cellular Neural Networks (CNN) are homogeneous arrays of identical and identically coupled cells. An interesting behavior CNN's can exhibit is that of pattern formation. One of the architectures able to produce patterns is based on second order two-port cells coupled by means of two resistive grids (Chua *et al.*, 1988; Goraş *et al.*, 1995). The specific feature of Turing patterns is that the isolated cells are stable while the dynamics of the array can exhibit unstable spatial modes. If the cells are piecewise linear, a powerful method of

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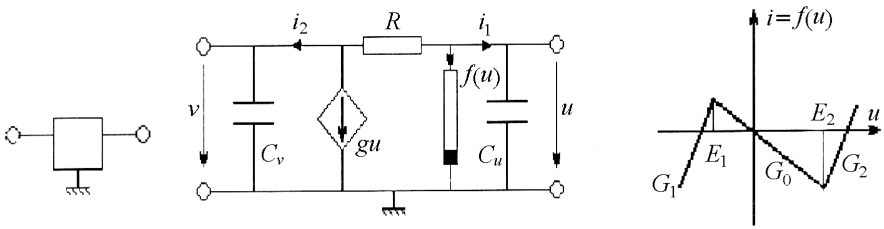
investigation represents the decoupling technique basically consisting of a change of variable chosen according to the boundary conditions. A conjecture regarding the limits of the characteristic polynomials roots is made and verified through simulations.

## 2. The Architecture of the Two-Grid Coupled CNN's and the Decoupling Technique

A possible realization of a piecewise linear cell and the architecture of the two-grid coupled CNN are represented in Fig.1.




*a* *b*

Fig. 1 – *a* – Two-port cell; *b* – piecewise linear characteristic of the nonlinear resistor.

The equations describing the CNN composed of *M* cells for the linear central part, ***i.e.*** for ***u***-voltages of the cell in Fig. 1 ***b*** within the interval [$E_1$, $E_2$], are (Goraş, 1995)

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = g\,(f_u u_i + f_v v_i) + D_u \nabla^2 u_i\,,$$
$$\frac{\mathrm{d}v_i(t)}{\mathrm{d}t} = g\,(g_u u_i + g_v v_i) + D_v \nabla^2 v_i\,,\ \ (i = 0,\ldots,M-1), \tag{1}$$

where $f_u$ , $f_v$ , $g_u$ , $g_v$ are the elements of the Jacobian matrix of $f(u,v)$ and $g(u,v)$, $D_u$ and $D_v$ – the diffusion coefficients, γ is a scaling coefficient and $\nabla^2 x_i$ stands for the 1-D Laplacean $\nabla^2 x_i = x_{i+1} + x_{i-1} - 2x_i$ . The relations between the above coefficients and the circuit elements of the CNN cell are

$$f_u = -(G + G_0),\ f_v = G,\ g_u = \frac{C_u}{C_v}(G - g),\ g_v = -\frac{C_u}{C_v}G\,. \tag{2}$$

The analysis of the CNN dynamic behavior can be simplified in the linear part of the cells characteristics using the decoupling technique (Goraş *et al.*, 1995; Goraş, 2002; Goraş *et al.*, 1996, 1997). We transform the system of equations by means of the change of variable

$$u_i(t) = \sum_{m=0}^{M-1} \Phi_M(i,m)\hat{u}_m(t),$$
$$v_i(t) = \sum_{m=0}^{M-1} \Phi_M(i,m)\hat{v}_m(t),\ (i = 0,\ldots,M-1), \tag{3}$$

where $\Phi_M(i,m)$ are eigenfunctions (depending on the boundary conditions) of the 1-D Laplacean, *i.e.* $\nabla^2\Phi_M(i,m) = -k_m^2\Phi_M(i,m)$ and $k_m^2$– the eigenvalues, proportional to the square (or sum of squares) of sine functions.

With the above change of variable, the set of 2$M$ coupled differential equations in the $u$ and $v$ variables is transformed into $M$ sets of decoupled pairs of second order linear differential equations in the new variables – the amplitudes of the spatial components of the voltages

$$\begin{bmatrix} \dot{\hat{u}}_m \\ \dot{\hat{v}}_m \end{bmatrix} = \left( g \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} - k_m^2 \begin{bmatrix} D_u & 0 \\ 0 & D_v \end{bmatrix} \right) \begin{bmatrix} \hat{u}_m \\ \hat{v}_m \end{bmatrix}. \tag{4}$$

For ring boundary conditions, the eigenvectors are $\Phi_M(i,m) =$ $= \exp(\mathrm{j}2p\,mi/M)$ and the eigenvalue is $k_m^2 = 4\sin^2\dfrac{mp}{mM}$.

The natural frequencies, $l_{m1}$ and $l_{m2}$, are the roots of the characteristic polynomials

$$l_m^2 + \left[k_m^2(D_u + D_v) - g(f_u + g_v)\right] l_m + D_u D_v k_m^4 - l(D_v f_u + D_u g_v)k_m^2 +$$
$$+ (f_u g_v - f_v g_u) = 0,\ (m = 0,\ldots,M-1).$$

The solutions of the 1-D CNN equations are thus

$$u_i(t) = \sum_{m=0}^{M-1} \left(a_m \mathrm{e}^{l_{m1}t} + b_m \mathrm{e}^{l_{m2}t}\right)\Phi_M(i,m),$$
$$v_i(t) = \sum_{m=0}^{M-1} \left(c_m \mathrm{e}^{l_{m1}t} + d_m \mathrm{e}^{l_{m2}t}\right)\Phi_M(i,m),\ (i = 0,\ldots,M-1), \tag{5}$$

where the constants $a_m$, $b_m$, $c_m$, $d_m$ depend on the initial conditions.

Considering the decoupling technique that has been presented above, the mechanism of the pattern formation, that is the existence of unstable spatial modes, can be explained by means of the existence of positive real values for the dispersion curve of the modes and also by means of the nonzero initial conditions of those modes,

$$\Re e\, l_{1,2}\left(k_m^2\right)=\Re e\left\{ l\,\frac{f_u+g_v}{2}-k_m^2\,\frac{D_u+D_v}{2}+\sqrt{\left[ l\,\frac{g_v-f_u}{2}+k_m^2\,\frac{D_u-D_v}{2}\right]^2+g^2 f_v g_u}\right\}. \quad (6)$$

In order to have an image concerning the manner in which each of the six parameters influences the dispersion curve (Goraş *et al.*, 2007), several graph families have been presented in Fig. 2 *a*,…,2 *f*.

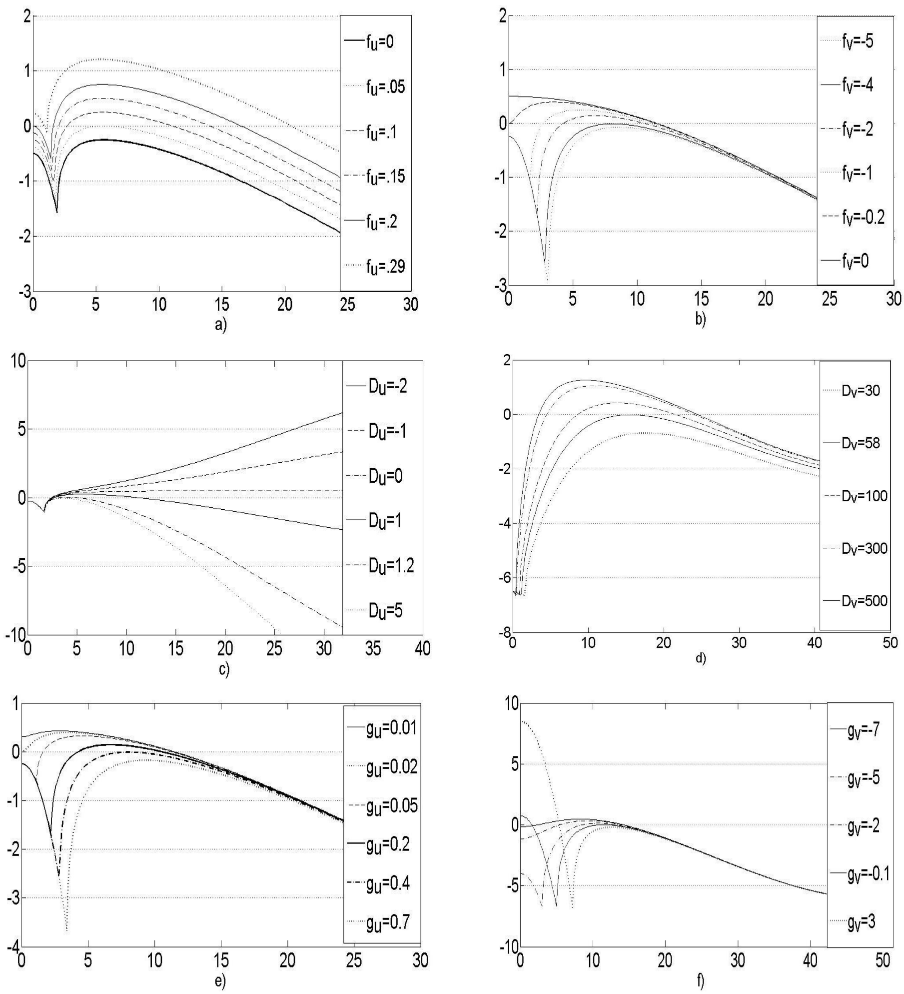



Fig. 2 – Family of dispersion curves for variable:
$a - f_u$, $b - f_v$, $c - D_u$, $d - D_v$, $e - g_u$, $f - g_v$.

The peak of the dispersion curve is located at the value (Goraş *et al.*, 1995)

$$k_p^2 = \left[ (g_v - f_u) + \frac{D_u + D_v}{\sqrt{D_u D_v}} \sqrt{-f_v g_u} \right] \frac{g}{D_v - D_u}. \tag{7}$$

It has been shown that using a spectral decoupling technique, valid for the linear part of the transient, the final pattern can be predicted to a more or less extent. The pattern formation may be regarded as a result of the competition between modes, their strengths and values being equally important.

## 3. Nonhomogeneous CNN's and Gershgorin's Circles

The main idea of this research is to investigate the behavior in terms of robustness of a two-grid coupled CNN when the paramereters vary. We want to see if the cell parameter variations within certain limits will preserve the existence of a band of unstable modes and in this way the instabillity of the array. For this purpose it will be used the Gershgorin circle theorem (Cvetkovic *et al*.,2004; Hote *et al*., 2006).

Gershgorin theorem gives bounds for the region in the complex plane where the eigenvalues of a matrix lie.

Given a square matrix of order *n*, $A = (a_{i,j}) \in C$ ,then all its eigenvalues lie in the union of *n* circles

$$\bigcup_{i=1}^{n} \left\{ z \in C : \left| z - a_{ii} \right| \le r_i(A) \right\} = \bigcup_{i=1}^{n} \Gamma_i(A) = \Gamma(A),$$

where $r_i(A) = \sum_{j=1, j \ne i}^{n} \left| a_{ij} \right|$ (sum of off-diagonal elements in row *i* ); $\Gamma_i(A)$ is the *i*-th Gershgorin circle and $\Gamma(A)$ – the Gershgorin set for *A*.

Noting that matrix *A* and its transpose $A^T$ have the same eigenvalues, applying the above theorem for $A^T$ yields another set $\hat{\Gamma}(A)$. The eigenvalues of matrix *A* are lying in the complex plane within the intersection of the Gershgorin sets $\Gamma(A)$ and $\hat{\Gamma}(A)$.

To simplify the notations, in the following we consider as an example a 5-cell CNN with ring boundary conditions which is described by the set of state equations of order 10 (each cell is of second order):

$$\begin{bmatrix}\dot{u}_0\\ \dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \dot{u}_4\\ \dot{v}_0\\ \dot{v}_1\\ \dot{v}_2\\ \dot{v}_3\\ \dot{v}_4\end{bmatrix}=\begin{bmatrix}
gf_u-2D_u & D_u & 0 & 0 & D_u & gf_v & 0 & 0 & 0 & 0\\
D_u & gf_u-2D_u & D_u & 0 & 0 & 0 & gf_v & 0 & 0 & 0\\
0 & D_u & gf_u-2D_u & D_u & 0 & 0 & 0 & gf_v & 0 & 0\\
0 & 0 & D_u & gf_u-2D_u & D_u & 0 & 0 & 0 & gf_v & 0\\
D_u & 0 & 0 & D_u & gf_u-D_u & 0 & 0 & 0 & 0 & gf_v\\
gg_u & 0 & 0 & 0 & 0 & gg_v-D_v & D_v & 0 & 0 & 0\\
0 & gg_u & 0 & 0 & 0 & D_v & gg_v-2D_v & D_v & 0 & 0\\
0 & 0 & gg_u & 0 & 0 & 0 & D_v & gg_v-2D_v & D_v & 0\\
0 & 0 & 0 & gg_u & 0 & 0 & 0 & D_v & gg_v-2D_v & D_v\\
0 & 0 & 0 & 0 & gg_u & D_v & 0 & 0 & D_v & gg_v-2D_v
\end{bmatrix}\cdot\begin{bmatrix}u_0\\ u_1\\ u_2\\ u_3\\ u_4\\ v_0\\ v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}$$

Using the above change of variable, the equations decouple and become

$$\begin{bmatrix}\dot{\hat{u}}_0\\ \dot{\hat{v}}_0\\ \dot{\hat{u}}_1\\ \dot{\hat{v}}_1\\ \dot{\hat{u}}_2\\ \dot{\hat{v}}_2\\ \dot{\hat{u}}_3\\ \dot{\hat{v}}_3\\ \dot{\hat{u}}_4\\ \dot{\hat{v}}_4\end{bmatrix}=\begin{bmatrix}
gf_u-k_0^2D_u & gf_v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
gg_u & gg_v-k_0^2D_v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & gf_u-k_1^2D_u & gf_v & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & gg_u & gg_v-k_1^2D_v & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & gf_u-k_2^2D_u & gf_v & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & gg_u & gg_v-k_2^2D_v & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & gf_u-k_3^2D_u & gf_v & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & gg_u & gg_v-k_3^2D_v & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & gf_u-k_4^2D_u & gf_v\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & gg_u & gg_v-k_4^2D_v
\end{bmatrix}\cdot\begin{bmatrix}\hat{u}_0\\ \hat{v}_0\\ \hat{u}_1\\ \hat{v}_1\\ \hat{u}_2\\ \hat{v}_2\\ \hat{u}_3\\ \hat{v}_3\\ \hat{u}_4\\ \hat{v}_4\end{bmatrix}$$

The difference between the two matrices that describe the dynamics of the array is that the latter implies the homogeneity of the whole array (Alecsandrescu *et al.*, 2008). It can be thus used only for homogeneous variations of the parameters. For each matrix we will determine two Gershgorin sets and we will compare the results of the two approaches in what concerns the rightmost circle whose position is related to the existence of a band of unstable modes.

Due to the symmetries of the array and thus of the equations, there are only two values for the Gershgorin circles for each CNN, only the radii being variable.

The centers of the Gershgorin circles for the first matrix are $c_1 = g\, f_u - 2D_u$; $c_2 = g\, g_v - 2D_v$ and the radii $r_1 = |2D_u| + |g\, g_u|$; $r_2 = |2D_v| + |g\, f_v|$ (computed for columns) and $r_1 = |2D_u| + |g\, f_v|$; $r_2 = |2D_v| + |g\, g_u|$ (computed for rows). For the matrix of the decoupled equations the centers of the Gershgorin circles are $c_1 = g\, f_u - k_m^2 D_u$; $c_2 = g\, g_v - k_m^2 D_v$ and the radii are $r_1 = |g\, f_v|$; $r_2 = |g\, g_u|$, for both centers.

## 4. Simulation Results

The following results have been obtained for a 1-D array of length $M = 10$ with periodic (ring) boundary conditions and characterized by $f_u = 0.4$, $f_v = -1$, $g_u = 1.5$, $g_v = -2$, $D_u = 2$, $D_v = 120$ and $\gamma = 10$.

Gershgorin circles centers are $c_1 = g\, f_u - 2D_u = 0$; $c_2 = g\, g_v - 2D_v =$ $= -260$ and the radii computed for columns and rows: $r_1 = |2D_u| + |g\, g_u| = 19$; $r_2 = |2D_v| + |g\, f_v| = 250$ and $r_1 = |2D_u| + |g\, f_v| = 14$; $r_2 = |2D_v| + |g\, g_u| = 255$, respectively.

For this example, the peak of the dispersion curve is $k_p^2 = 0.614$ according to (7) and $\Re el\left(k_p^2\right) = 1.19$.

From above values it can be observed that the rightmost abscissa for the eigenvalues is 19 and is larger than the greatest real part of the eigenvalues so that there is no reason to state that the array will be unstable. Using the matrix for decoupled equations the Gershgorin centers and radii for the adopted values of the parameters are given by the core matrix

$$\begin{bmatrix} 4 - 2k_m^2 & -10 \\ 15 & -20 - 120k_m^2 \end{bmatrix},$$

where $k_m^2 \in [0,4]$. Thus, the abscissas of centers of the Gershgorin circles are $4-k_m^2$ and $-20-120k_m^2$, ***i.e.*** they have values between – 4 and 4 and between – 460 and – 20. The radii are 15 and 10. The rightmost absissa of the circles is 19 as in the previous case.

In what follows we vary some of the CNN parameters, ***i.e.*** *γ*, $D_v$, $f_u$, $g_v$, $g_u$; the associated Gershgorin circles have been represented in Figs. 3,…,7. From these figures it can be seen that many of the circles have no influence on the stability limits.

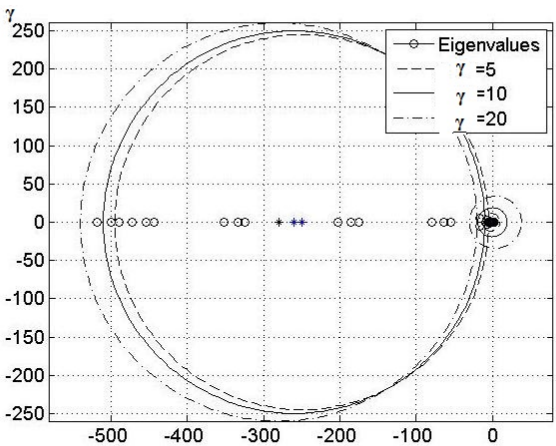



Fig. 3 – Gershgorin circles for *γ* parameter variation.

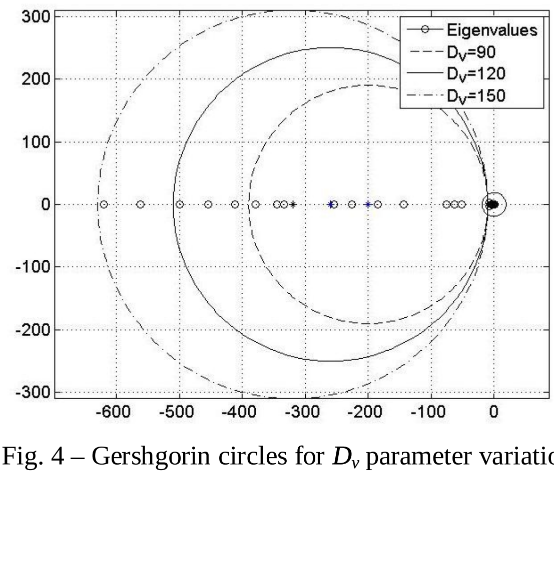



Fig. 4 – Gershgorin circles for $D_v$ parameter variation.

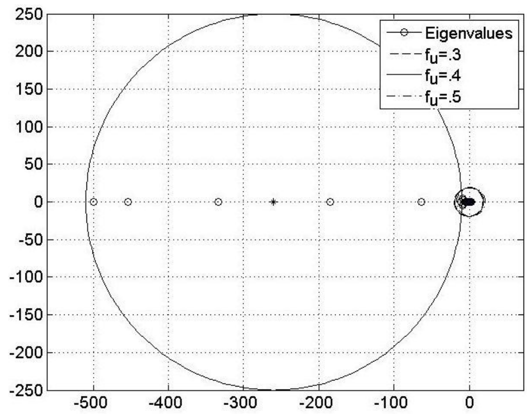



Fig. 5 – Gershgorin circles for $f_u$ parameter variation.

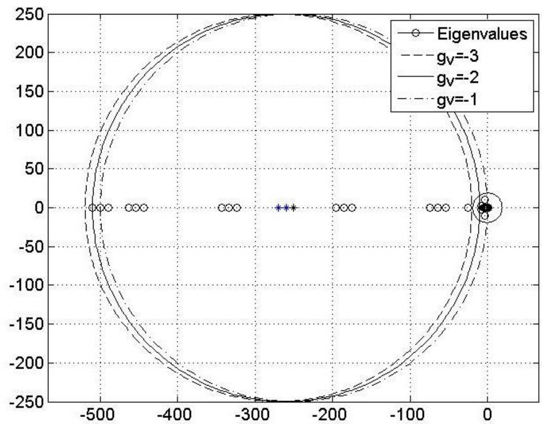



Fig. 6 – Gershgorin circles for $g_v$ parameter variation.

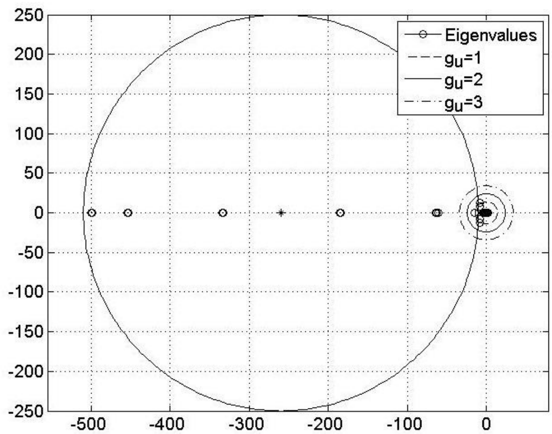



Fig. 7 – Gershgorin circles for $g_u$ parameter variation.

## 5. Conclusions

In this paper we have investigated how the parameters variations of a double grid second order cell CNN within certain limits can influence the robustness of the network. It has been shown that the stability limits estimated using the theory of Gershgorin circles are larger than expected according to simulations. These limits define the frontier of the domain for preserving the instability of the network.

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ROBUSTEŢEA REŢELELOR NEURONALE CELULARE LA VARIAŢIA PARAMETRILOR

(Rezumat)

Se prezintă rezultatele investigării robusteţei Reţelelor Neuronale Celulare (RNC) pentru evoluţii în zona central liniară la variaţia parametrilor. Rezultatele

simulărilor confirmă faptul că variaţiile parametrilor celulei între anumite limite stabilite nu modifică caracterul instabil al reţelei. Instrumentul matematic folosit în acest scop îl constituie teorema cercurilor lui Gershgorin. Acestea furnizează limitele pentru valorile proprii, cel mai din dreapta cerc având legătură cu existenţa benzii de moduri instabile. Se constată că aceste cercuri conduc la limite mai mari pentru valorile proprii asftel încât trebuie utilizate cu grijă.