BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Tomul LVII (LXI), Fasc. 6, 2011
Secţia
ELECTROTEHNICĂ. ENERGETICĂ. ELECTRONICĂ

THE HAMILTON-POISSON REALIZATION OF THE SEPARATOR MATHEMATICAL MODEL

BY

ANANIA ARON[*] and ŞTEFAN KILYENI

"Politehnica" University of Timişoara

Received, May 31, 2011
Accepted for publication: July 29, 2011

Abstract. A particular case of the separator system of differential equations is considered, realizing this system as a Hamiltonian one and then the system is studied from some standard Poisson geometry points of view. The exact solution of the considered differential system is provided through the obtained Hamilton-Poisson realization and thus a comprehensive system analysis can be performed through standard Poisson geometry points of view.

Key words: differential equations; Hamilton-Poisson geometry; energy, battery mathematical model; constant of motion.

1. Introduction

In the last decades, a continuous trend of miniaturization of electronic devices has been observed as well as a diversification in terms of mobility, functionalities and communication. In this context, there is also a diversification of the battery types used as energy source, a unified approach of their behavior based on energy consumption being a difficult task as there are constraints related to the nonlinear battery behavior caused by the rate capacity effect, relaxation effect, capacity fading, self discharge and temperature influence. Due to these aspects, several batteries models are in place, based on the specific constraints of an electronic device: the linear battery model, the electrochemical

[*]Corresponding author: *e-mail*: anania.girban@gmail.com

models, and some types of models based on a trade-off between accuracy and required computational effort.

The battery mathematical models are used in simulators, emulators and numerical systems that monitors the batteries status during systems runtime, in general, having as purpose to collect information about the batteries state-of-charge and batteries behavior on different load profiles and different critical situations. Among the target systems where this research is going to be applied we can enumerate the nodes of wireless sensor networks, monitoring the batteries of backup sources for electrical power substation ancillary services as well as in automotive industry where an active research area is prolonging the batteries life.

This paper presents a geometrical approach able to provide more comprehensive information about dynamic systems than traditional methods currently used in battery modeling. The Li-Ion electrochemical model is used as reference (Subramanian *et al.*, 2009).

We are considering here a particular case of the separator system, realizing this system as a Hamiltonian one and then study the system from some standard Poisson geometry points of view. The most accurate mathematical modeling of Li-Ion batteries are considered in the literature to be the one presented by Subramanian *et al.* (2009). In this paper we discuss the separator differential eqs. In the literature, all other approaches are based on the numerical integration. If we are able to find a Hamilton-Poisson realization, that gives us the exact solution of the system. The separator eqs. given by Subramanian *et al.* (2009) after some transformations can be put into the following form on $\mathbb{R}^3$:

$$\begin{cases} x_1'(q) = x_2(q), \\ x_2'(q) = a\, x_2(q), \\ x_3'(q) = b\dfrac{x_2(q)}{x_1(q)} + d, \end{cases}$$

where *a, b, d* are real parameters.

We investigate the possibility to give a Hamilton-Poisson realization of the corresponding system. For certain values of the parameters, the existence of one functionally independent Casimir function of the corresponding Hamilton-Poisson realization is shown. The next step is to find the Casimir functions of the configuration. Since the Poisson structure is degenerate, we can try to obtain Casimir functions *via* the algebraic method of Bermejo-Fairen (see Hernandez-Bermejo & Fairen (1998)).

2. Geometrical Approach of the Differential Model

Further on, we will refer to the governing equations of a Li-Ion battery model which is also depicted as a *pseudo-two-dimensional isothermal model*.

We consider the separator eqs. for liquid phase transport and the modified Ohm's law for potential distribution and obtain the next three dimensional differential system after some transformations in which we consider $x_1 = c$, $x_2 = c'$, $x_3 = \Phi_2$, $q = ux + t$,

$$\begin{cases} x_1'(q) = x_2(q), \\ x_2'(q) = a\,x_2(q), \\ x_3'(q) = b\dfrac{x_2(q)}{x_1(q)} + d, \end{cases} \tag{1}$$

where *a, b, d, u* are real constants.

We investigate the possibility to realize a Hamilton-Poisson realization of the corresponding system and the existence of one functionally independent Casimir function of the corresponding Hamilton-Poisson realization.

2.1. The Poisson Geometry Associated to a Smooth Linear Version of a Lithium-Ion Battery Model System

P r o p o s i t i o n 1. *The following smooth real function, H, is constant of the motion defined by the system* (1),

$$H(x_1, x_2, x_3) = ax_1 - x_2,$$

where $a \in \mathbb{R}$.

P r o o f. It is easy to see that d$H = 0$.

The goal of this section is to try to find a Hamilton-Poisson structure for system (1). We shall consider the skew-symmetric matrix given by

$$\Pi = \begin{bmatrix} 0 & p_1(x_1,x_2,x_3) & p_2(x_1,x_2,x_3) \\ -p_1(x_1,x_2,x_3) & 0 & p_3(x_1,x_2,x_3) \\ -p_2(x_1,x_2,x_3) & -p_3(x_1,x_2,x_3) & 0 \end{bmatrix}.$$

We have to find the real smooth functions $p_1, p_2, p_3 : \mathbb{R}^3 \to \mathbb{R}$, such that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \Pi \cdot \nabla H$$

and

$$-p_3\left(\frac{\partial p_2}{\partial x_3}+\frac{\partial p_1}{\partial x_2}\right)+p_2\left(\frac{\partial p_3}{\partial x_3}+\frac{\partial p_1}{\partial x_1}\right)+p_1\left(\frac{\partial p_3}{\partial x_2}+\frac{\partial p_2}{\partial x_1}\right)=0. \tag{2}$$

This is the Jacobi identity and it should be satisfied. From this condition we obtain the following differential eq.:

$$bx_2^2-ax_1^2p_3+x_1(dx_1+bx_2)\frac{\partial p_3}{\partial x_3}+a\,x_1^2x_2\frac{\partial p_3}{\partial x_2}+x_1^2x_2\frac{\partial p_3}{\partial x_1}=0. \tag{3}$$

Under the above assumptions (2) and (3) have the following solutions:

$$\begin{cases} p_1(x_1,x_2,x_3)=-x_2, \\ p_2(x_1,x_2,x_3)=\dfrac{-d+x_2\varphi(u,v)}{a}, \\ p_3(x_1,x_2,x_3)=\dfrac{x_2(b+x_1\varphi(u,v))}{x_1}, \end{cases} \tag{4}$$

where $\varphi\in C^1(\mathbb{R}^2)$ is an arbitrary real function and

$$u=-ax_1+x_2\,,\qquad v=x_3-b\ln(x_1)-\frac{d\ln(x_2)}{a}. \tag{5}$$

Then, we obtain the next result:

P r o p o s i t i o n 2. *The system* (*1*) *has the Hamilton-Poisson realization*

$$(\mathbb{R}^3,\ \Pi{:=}[\ \Pi^{ij}\ ],\ H),$$

where

$$\Pi=\begin{bmatrix} 0 & p_1(x_1,x_2,x_3) & p_2(x_1,x_2,x_3) \\ -p_1(x_1,x_2,x_3) & 0 & p_3(x_1,x_2,x_3) \\ -p_2(x_1,x_2,x_3) & -p_3(x_1,x_2,x_3) & 0 \end{bmatrix},$$

with $p_i(x_1, x_2, x_3)$, (i = 1, 2, 3), *given in (4),* $\varphi\in C^1(\mathbb{R}^2)$, $\varphi=\varphi(u,v)$, *where u, v are given in (5), is an arbitrary real function, and*

$$H(x_1,x_2,x_3)=ax_1-x_2\,,\quad a\in\mathbb{R}.$$

The next step is to try to find the Casimir functions of the configuration described by the Proposition 2 in the case when $\varphi(u,v) = 0$, for any u, $v \in \sqcup$. Then the matrix Π becomes

$$\Pi = \begin{bmatrix} 0 & p_1(x_1,x_2,x_3) & p_2(x_1,x_2,x_3) \\ -p_1(x_1,x_2,x_3) & 0 & p_3(x_1,x_2,x_3) \\ -p_2(x_1,x_2,x_3) & -p_3(x_1,x_2,x_3) & 0 \end{bmatrix},$$

where

$$p_1(x_1,x_2,x_3) = -x_2\,, \quad p_2(x_1,x_2,x_3) = \frac{-d}{a}\,, \quad p_3(x_1,x_2,x_3) = \frac{x_2 b}{x_1}. \tag{6}$$

The Poisson structure is degenerated, so we are looking for existence of Casimir functions. The defining equations for the Casimir functions, denoted by C, are $\Pi^{ij}\,\partial_j\, C = 0$.

In this case a finite dimensional Hamilton-Poisson system, the determination of a Casimir function could be performed *via* the algebraic method of Bermejo-Fairen (1998). Since the rank of Π is constant and equal to 2, there exists only one functionally independent Casimir function associated to our structure. The Bermejo-Fairen technique consists of two steps: we compute explicitly the components of the matrix given by

$$\Gamma = \left(\Pi_1 \cdot \Pi_2^{-1}\right)^t,$$

where

$$\Pi_1 = \left[\, -p_2(x_1,x_2,x_3) \quad -p_3(x_1,x_2,x_3) \right]$$

and

$$\Pi_2 = \begin{bmatrix} 0 & p_1(x_1,x_2,x_3) \\ -p_1(x_1,x_2,x_3) & 0 \end{bmatrix}.$$

We obtain

$$\Gamma = \begin{bmatrix} -\dfrac{p_3(x_1,x_2,x_3)}{p_1(x_1,x_2,x_3)} \\ \dfrac{p_2(x_1,x_2,x_3)}{p_1(x_1,x_2,x_3)} \end{bmatrix}.$$

One associates the Pffafian system

$$\mathrm{d}x_3 = \Gamma_1 \mathrm{d}x_1 + \Gamma_2 \mathrm{d}x_2,$$

i.e.

$$\mathrm{d}x_3 = -\frac{p_3(x_1,x_2,x_3)}{p_1(x_1,x_2,x_3)}\mathrm{d}x_1 + \frac{p_2(x_1,x_2,x_3)}{p_1(x_1,x_2,x_3)}\mathrm{d}x_2,$$

where p_1, p_2, p_3 are given by the formulas (6). For the integrability of this Pffafian system (Yudichak *et al*., 1999) one needs an integrand factor to transform it into an equivalent one such that the above form is exact. The existence of an integrant factor is guaranteed by the Frobenius Theorem (Yudichak *et al*., 1999). One obtains a Casimir function of our configuration given by the following expression:

$$C(x_1,x_2,x_3) = \mathrm{e}^{x_3}\, x_1^{-b}\, x_2^{-d/a}.$$

Consequently we have derived the following result:
P r o p o s i t i o n 3. *The real smooth function* $C: \mathbb{R}^3 \to \mathbb{R}$,

$$C(x_1,x_2,x_3) = \mathrm{e}^{x_3}\, x_1^{-b}\, x_2^{-d/a}$$

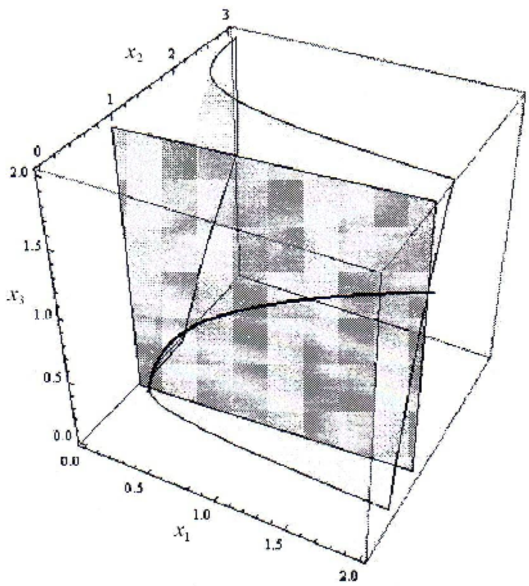

Fig. 1 – The phase curve of the system (1).

is the only one functionally independent Casimir function of the Hamilton-Poisson realization

$$\left(R^3, \Pi := \begin{bmatrix} 0 & p_1 & p_2 \\ -p_1 & 0 & p_3 \\ -p_2 & -p_3 & 0 \end{bmatrix}, H \right)$$

of the linear differential system (*1*), *where* p_1, p_2, p_3, H: $\mathbb{R}^3 \to \mathbb{R}$, *are, respectively, given by* (*6*),

$$H(x_1, x_2, x_3) = ax_1 - x_2, \ a \in \mathbf{R}.$$

In the Fig.1 we have sketched the phase curves of the dynamics (1). The phase curves are the intersections of the surfaces: H = const. and C = const.

3. Conclusions

The obtained results are partial solutions and also going to be experimentally confirmed.

The results are interesting from the mathematical point of view, which will be correlated with the results obtained for the two electrodes in order to be integrated in a battery mathematical model to be used in WSN simulators and online monitoring of the wireless sensors batteries. Other applications of these results are the monitoring of the batteries in backup sources for electrical power substation ancillary services and, perhaps, the batteries used in automotive industry.

The next step in our work is to study the stability of the system using the specific methods from Hamilton-Poisson geometry (Birtea & Puta, 2007; Tudoran *et al*., 2009), completing the well known methods (Arnold, 1965). Since we were able to find the analytic solution of the system (1) it would be interesting to compare the found solution with the solution given by the numerical integration, in particular with the Runge Kutta integrator as well as with the Kahan integrator and analyse the differences. The Kahan integrator gives simplest and more efficient methods which are adequate to the proposed scope (better performances related to the validated solution, the analytic solution, used as reference).

Acknowledgment. This paper was supported by the project "Development and Support of Multidisciplinary Postdoctoral Programmes in Major Technical Areas of National Strategy of Research – Development – Innovation" 4D-POSTDOC, contract no. POSDRU/89/1.5/S/52603, project co-funded by the European Social Fund through Sectoral Operational Programme Human Resources Development 2007-2013.

REFERENCES

Arnold V., *Conditions for Nonlinear Stability of Stationary Plane Curvilinear Flows of an Ideal Fluid*. Dokl. Akad. Nauk., URSS, **162**, *5*, 773-777 (1965).
Birtea P., Puta M., *Equivalence of Energy Methods in Stability Theory*. J. of Math. Phys., **48**, *4*, 81-99 (2007).
Hernandez-Bermejo B., Fairen V., *Simple Evaluation of Casimirs Invariants in Finite Dimensional Poisson Systems*. Phys. Lett. A, *241*, 148-154 (1998).
Puta M., Tudoran R.M., Tudoran R.M., *Poisson Manifolds and Bermejo-Fairen Construction of Casimirs*. Tensor N.S., **66**, 59-70 (2005).
Subramanian V.R., Boovaragavan V., Ramadesigan V., Arabandi M., *Mathematical Model Reformulation for Lithium-Ion Battery Simulations: Galvanostatic Boundary Conditions*. J. of The Electrochem. Soc., **156**, *4*, 260-271 (2009).
Tudoran R.M., Aron A., Nicoara S., *On a Hamiltonian Version of the Rikitake System*. SIAM J. Appl. Dyn. Syst., **8**, *1*, 454-479 (2009).
Yudichak T.W., Hernandez-Bermejo B., Morrison P.J., *Competing Invariants from Pffafian Systems*. Phys. Lett. A, *260*, 475-483 (1999).

REALIZAREA HAMILTON-POISSON A MODELULUI MATEMATIC AL SEPARATORULUI

(Rezumat)

Se consideră un caz particular al sistemului de ecuaţii diferenţiale ce modelează separatorul, ca sistem Hamiltonian, şi apoi se studiază sistemul din câteva puncte de vedere standard ale geometriei Poisson standard. Soluţia exactă a sistemului diferenţial considerat este obţinută prin intermediul realizării Hamilton-Poisson găsite şi în acest fel se poate efectua o analiză completă a sistemului din punctul de vedere al geometriei Poisson standard.