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ON SOME NEW RECIPROCITY RELATIONS IN THE ELECTROMAGNETIC FIELD

BY

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Abstract. Some new reciprocity relations in an electromagnetic field evolving in a motionless, linear, isotropic, homogeneous or non-homogeneous, with or without losses medium, are established.

Key words: electromagnetic field; electromagnetic induction low; Lorentz calibration relation; vectorial identities; reciprocity relations.

1. Introduction

Reciprocity relations in an electromagnetic field were initially established by Lorentz (1895) and then extended by numerous research workers among which may be quoted Edm. Nicolau (1953, 1954) and E. Lugaresi (1957). These relations, using Maxwell’s eqs., involve state vectors **E** and **H** of an electromagnetic field as well as the material constants of this field, ε, μ, σ. In two previous papers (Rosman, 2007, 2008) such reciprocity relations were obtained associating Maxwell equations with some vectorial identities.

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The aim of this paper is to render evident the possibility to obtain other reciprocity relations associating the local form of electromagnetic induction low as well as Lorentz calibration relation with some vectorial indentities. In this case, in reciprocity relations intervene other quantities than state vectors **E** and **H**. More precisely in the studied case, the scalar and vectorial potentials intervene in the new reciprocity relations.

2. Utilized Equations

With the view of obtaining the new reciprocity relations, two different states of a same electromagnetic field are considered, characterized by the scalar potentials V_1, V_2 and the vectorial ones, $\mathbf{A}_1$, $\mathbf{A}_2$. In the same time are utilized: a) the local form of electromagnetic induction law

$$\mathbf{E}_1 = -\mathrm{grad}V_1 - \frac{\partial \mathbf{A}_1}{\partial t},\ \ \mathbf{E}_2 = -\mathrm{grad}V_2 - \frac{\partial \mathbf{A}_2}{\partial t}, \tag{1}$$

and b) Lorentz calibration relation (Mocanu, 1981)

$$\mathrm{div}\mathbf{A}_1 = -\varepsilon\mu\frac{\partial V_1}{\partial t},\ \ \mathrm{div}\mathbf{A}_2 = -\varepsilon\mu\frac{\partial V_2}{\partial t}, \tag{2}$$

the notations being the usuals ones.

The electromagnetic field is supposed variable, evolving in a motionless, linear, isotropic, homogeneous or non-homogeneous medium, with or without losses.

To eqs. (1) and (2), the vectorial identities (Beju *et al*., 1996)

$$\mathrm{div}\left(M\mathbf{N}\right) = M\ \mathrm{div}\ \mathbf{N} + \mathbf{N}\ \mathrm{grad}M, \tag{3}$$

$$\mathrm{rot}\left(M\mathbf{N}\right) = M\ \mathrm{rot}\ \mathbf{N} - \mathbf{N}\times\mathrm{grad}M \tag{4}$$

are associated.

The reciprocity relations established in the following section of this paper are obtained performing in the vectorial identities (3), successively, the substitutions

$$M{=}V_1,\ \ \mathbf{N}{=}\mathbf{A}_2;\ \ M{=}V_2,\ \ \mathbf{N}{=}\mathbf{A}_1. \tag{5}$$

3. Establishing of the Reciprocity Relations

3.1. First Reciprocity Relation

If the substitutions (5) are performed, successively, in vectorial identity (3) and taking in account relations (1) and (2) it results

$$\operatorname{div}\left(V_1\mathbf{A}_2\right) = -\varepsilon\mu V_1\frac{\partial V_2}{\partial t} - \mathbf{A}_2\left(\mathbf{E}_1 + \frac{\partial \mathbf{A}_1}{\partial t}\right), \tag{6}$$

respectively

$$\operatorname{div}\left(V_2\mathbf{A}_1\right) = -\varepsilon\mu V_2\frac{\partial V_1}{\partial t} - \mathbf{A}_1\left(\mathbf{E}_2 + \frac{\partial \mathbf{A}_2}{\partial t}\right). \tag{7}$$

The addition of relations (6) and (7) leads to

$$\operatorname{div}\left(V_1\mathbf{A}_2 + V_2\mathbf{A}_1\right) = -\frac{\partial}{\partial t}\left(\varepsilon\mu V_1 V_2 + \mathbf{A}_1\mathbf{A}_2\right) - \mathbf{A}_1\mathbf{E}_2 - \mathbf{A}_2\mathbf{E}_1 \tag{8}$$

which represents the local form of the first reciprocity relation established in this paper. The global form may be obtained by integration of relation (8) in the volume v_S bounded by the closed surface S

$$\iint_S\left(V_1\mathbf{A}_2 + V_2\mathbf{A}_1\right)\mathbf{dA} = -\frac{\partial}{\partial t}\iiint_{v_S}\left(\varepsilon\mu V_1 V_2 + \mathbf{A}_1\mathbf{A}_2\right)\mathrm{d}v - \iiint_{v_S}\left(\mathbf{A}_1\mathbf{E}_2 - \mathbf{A}_2\mathbf{E}_1\right)\mathrm{d}v. \tag{9}$$

Evidently, a reciprocity relation is obtained too if the difference of relations (6) amd (7) are performed, resulting his local form. The global form may be obtained using the same proceeding as in the previous case.

In the particular case when the variable state of the electromagnetic field is permanent harmonic, using the complex symbolic method, relations (6) and (7) may be writen

$$\operatorname{div}\left(V_1\underline{\mathbf{A}}_2\right) = -\mathrm{j}\omega\underline{\varepsilon}\underline{\mu}\underline{V}_1\underline{V}_2 - \underline{\mathbf{A}}_2\left(\underline{\mathbf{E}}_1 + \mathrm{j}\omega\underline{\mathbf{A}}_1\right), \tag{10}$$

respectively

$$\operatorname{div}\left(V_2\underline{\mathbf{A}}_1\right) = -\mathrm{j}\omega\underline{\varepsilon}\underline{\mu}\underline{V}_1\underline{V}_2 - \underline{\mathbf{A}}_1\left(\underline{\mathbf{E}}_2 + \mathrm{j}\omega\underline{\mathbf{A}}_2\right). \tag{11}$$

This time the reciprocity relation (local form) obtained performing the addition of relations (10) and (11) is less interesting. More interesting is the local form of the reciprocity relation if the difference of relations (10) and (11) is performed namely

$$\operatorname{div}\left(V_1\mathbf{A}_2 - V_2\mathbf{A}_1\right) = -\mathbf{E}_1\mathbf{A}_2 + \mathbf{E}_2\mathbf{A}_1, \tag{12}$$

the global form being

$$\iint_S\left(V_1\mathbf{A}_2 - V_2\mathbf{A}_1\right)\mathbf{dA} = \iiint_{v_S}\left(-\mathbf{E}_1\mathbf{A}_2 + \mathbf{E}_2\mathbf{A}_1\right)\mathrm{d}v. \tag{13}$$

The last two relations are independent with respect to the frequency.

It is necessary to specify that, in this case, the material constants are complex quantities (Mocanu, 1981)

$$\underline{\varepsilon}=\varepsilon'-\mathrm{j}\varepsilon'',\ \ \underline{\mu}=\mu'-\mathrm{j}\mu'', \tag{14}$$

the considered medium being a lossy one.

Because the material constants (real or complex) not intervene in relations (12) and (13); it results that these relations are valid both in case of a lossy medium and in that of a medium without losses.

3.2. Second Reciprocity Relation

In view to obtain this relation the substitutions (5) are performed in vectorial identity (4) resulting

$$\mathrm{rot}\left(V_1\mathbf{A}_2\right)=V_1\mathbf{B}_2+\mathbf{A}_2\times\left(\mathbf{E}_1+\frac{\partial\mathbf{A}_2}{\partial t}\right), \tag{15}$$

respectively

$$\mathrm{rot}\left(V_2\mathbf{A}_1\right)=V_2\mathbf{B}_1+\mathbf{A}_1\times\left(\mathbf{E}_2+\frac{\partial\mathbf{A}_1}{\partial t}\right), \tag{16}$$

where besides expressions (1) were taken into account the known relations (Mocanu, 1981)

$$\mathrm{rot}\mathbf{A}_1=\mathbf{B}_1,\ \ \mathrm{rot}\mathbf{A}_2=\mathbf{B}_2. \tag{17}$$

Performing the difference of relations (15) and (16) the expression

$$\mathrm{rot}\left(V_1\mathbf{A}_2-V_2\mathbf{A}_1\right)=V_1\mathbf{B}_2-V_2\mathbf{B}_1+\left(\mathbf{A}_2\times\mathbf{E}_1\right)-\left(\mathbf{A}_1\times\mathbf{E}_2\right)+\frac{\partial}{\partial t}\left(\mathbf{A}_1\times\mathbf{A}_2\right) \tag{18}$$

results representing the local form of the second reciprocity relation. Its global form may be obtained integrating relation (18) on the open surface S_l bounded by the closed curve l

$$\begin{aligned}\int_l\left(V_1\mathbf{A}_2-V_2\mathbf{A}_1\right)\mathbf{dl}=&\iint_{S_l}\left(V_1\mathbf{B}_2-V_2\mathbf{B}_1+\mathbf{A}_2\times\mathbf{E}_1-\mathbf{A}_1\times\mathbf{E}_2\right)\mathbf{dA}+\\&+\frac{\partial}{\partial t}\iint_{S_l}\left(\mathbf{A}_1\times\mathbf{A}_2\right)\mathbf{dA}.\end{aligned} \tag{19}$$

O b s e r v a t i o n. Having in view that div rot ≡ 0 it results

$$\mathbf{C}_1 = V_1 \mathbf{B}_2 + \mathbf{A}_2 \times \left(\mathbf{E}_1 + \frac{\partial \mathbf{A}_1}{\partial t} \right), \quad \mathbf{C}_2 = V_2 \mathbf{B}_1 + \mathbf{A}_1 \times \left(\mathbf{E}_2 + \frac{\partial \mathbf{A}_2}{\partial t} \right), \tag{20}$$

are of solenoidal type (div $\mathbf{C}_1 = 0$, div $\mathbf{C}_2 = 0$).

As in previous case (s. § **3.1**) a reciprocity relation may be determined adding relations (15) and (16). The obtained relation is less interesting.

In harmonic steady-state the complex expression of relations (15) and (16) are

$$\mathrm{rot}\left(\underline{V}_1 \underline{\mathbf{A}}_2\right) = \underline{V}_1 \underline{\mathbf{B}}_2 + \underline{\mathbf{A}}_2 \times \left(\underline{\mathbf{E}}_1 + \mathrm{j}\omega \underline{\mathbf{A}}_1\right), \tag{21}$$

respectively

$$\mathrm{rot}\left(\underline{V}_2 \underline{\mathbf{A}}_1\right) = \underline{V}_2 \underline{\mathbf{B}}_1 + \underline{\mathbf{A}}_1 \times \left(\underline{\mathbf{E}}_2 + \mathrm{j}\omega \underline{\mathbf{A}}_2\right), \tag{22}$$

so that their sum is

$$\mathrm{rot}\left(\underline{V}_1 \underline{\mathbf{A}}_2 + \underline{V}_2 \underline{\mathbf{A}}_1\right) = \underline{V}_1 \underline{\mathbf{B}}_2 + \underline{V}_2 \underline{\mathbf{B}}_1 + \underline{\mathbf{A}}_2 \times \underline{\mathbf{E}}_1 + \underline{\mathbf{A}}_1 \times \underline{\mathbf{E}}_2. \tag{23}$$

Expression (23) may be considered as representing the local form of the second reciprocity relation in harmonic steady state, utilizing the symbolic complex method. As regards the global form this one is

$$\int_l \left(\underline{V}_1 \underline{\mathbf{A}}_2 + \underline{V}_2 \underline{\mathbf{A}}_1\right) \mathbf{dl} = \iint_{S_l} \left(\underline{V}_1 \underline{\mathbf{B}}_2 + \underline{V}_2 \underline{\mathbf{B}}_1 + \underline{\mathbf{A}}_2 \times \underline{\mathbf{E}}_1 + \underline{\mathbf{A}}_1 \times \underline{\mathbf{E}}_2\right) \mathbf{dA}. \tag{24}$$

Relations (23) and (24) are independent with respect to the frequency.

Another reciprocity relation in harmonic steady-state may be obtained performing the difference of expressions (21) and (22) but the resulted relation is less interesting.

Having in view that in relations (18), (19), (23), (24) not intervene explicitely the material constants it results that these relations are valid both for lossy and without losses mediums.

In this case the complex vectors

$$\underline{\mathbf{C}}_1 = \underline{V}_1 \underline{\mathbf{B}}_2 + \underline{\mathbf{A}}_2 \times \underline{\mathbf{E}}_1 + \mathrm{j}\omega\left(\underline{\mathbf{A}}_2 \times \underline{\mathbf{A}}_1\right), \quad \underline{\mathbf{C}}_2 = \underline{V}_2 \underline{\mathbf{B}}_1 + \underline{\mathbf{A}}_1 \times \underline{\mathbf{E}}_2 + \mathrm{j}\omega\left(\underline{\mathbf{A}}_1 \times \underline{\mathbf{A}}_2\right) \tag{25}$$

and, evidently, their sum

$$\underline{\mathbf{C}} = \underline{\mathbf{C}}_1 + \underline{\mathbf{C}}_2 = \underline{V}_1 \underline{\mathbf{B}}_2 + \underline{V}_2 \underline{\mathbf{B}}_1 + \underline{\mathbf{A}}_2 \times \underline{\mathbf{E}}_1 + \underline{\mathbf{A}}_1 \times \underline{\mathbf{E}}_2, \tag{26}$$

are of solenoidal type.

4. Conclusions

Associating the vectorial identities (3), (4) with eqs. (1), (2) and (17) of an electromagnetic field, the local forms (8) and (18), and the global forms (9) and (19) of new reciprocity relations were established, available in variable state, when the electromagnetic field evolves in a motionless linear, isotropic, homogeneous or non-homogeneous, lossy or without losses medium. In the particular case when it is a matter of a harmonic steady-state, the reciprocity relations in complex form ((12), (13), respectively (23), (24)) are obtained.

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ASUPRA UNOR RELAȚII DE RECIPROCITATE ÎN CÂMPUL ELECTROMAGNETIC

(Rezumat)

Se stabilesc noi relații de reciprocitate într-un câmp electromagnetic variabil, care evoluează într-un mediu imobil, liniar, izotrop, omogen sau neomogen, cu sau fără pierderi. Relațiile de reciprocitate stabilite, spre deosebire de cele studiate în literatură, care implică vectorii de stare **E**, **H** ai câmpului, se referă la potențialul scalar, V şi la cel vector, **A**.