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THE COMPLEX TRANSFER IMPEDANCE OF A LINEAR TWO-PORT WITH NON-LINEAR RECEIVER

BY

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Abstract. The nonlinear differential equation of first order is established, satisfied by the function $X_2(R_2)$, where $\underline{Z}_2 = R_2 + jX_2$ is the complex impedance of the nonlinear inertial and passive receiver of a linear and non-autonomous in restreint sens two-port, in harmonic steady-state, so that the complex transfer impedance of the two-port have an extreme value.

The established differential equation is integrated analytically in two particular cases, when this one is of Bernoully type.

Key words: in restreint sense linear and non-autonomous two-ports; nonlinear inertial and passive receiver; complex transfer impedance; nonlinear differential equation of first order.

1. Introduction

It is well known that in the theory of the linear and non-autonomous in restreint sens two-ports, having the eqs.

$$\begin{bmatrix} \underline{U}_1 \\ \underline{I}_1 \end{bmatrix} = \left[\underline{A}\right] \begin{bmatrix} \underline{U}_2 \\ \underline{I}_2 \end{bmatrix}, \qquad (1)$$

with

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$$[\underline{A}]=\begin{bmatrix}\underline{A}_{11} & \underline{A}_{12}\\ \underline{A}_{21} & \underline{A}_{22}\end{bmatrix} \tag{2}$$

– the fundamental parameters matrix, may be defined, anmeng other transfer coefficients (Şora, 1964), the *transfer complex impedance*

$$\underline{Z}_m=\frac{\underline{U}_2}{\underline{I}_1}=R_m+\mathrm{j}X_m. \tag{3}$$

In what follows a detailed study of this complex impedance is performed, when the two-port's receiver is a nonlinear inertial one. It is a matter of a linear non-autonomous (in restreint sense) two-port (LNT), supplied at the (*1*), (*1′*) gate with a harmonic voltage having a nonlinear inertial and passive receiver (NIPR) represented in Fig. 1. The receiver's complex impedance is

$$\underline{Z}_2=\frac{\underline{U}_2}{\underline{I}_2}=R_2+\mathrm{j}X_2. \tag{4}$$

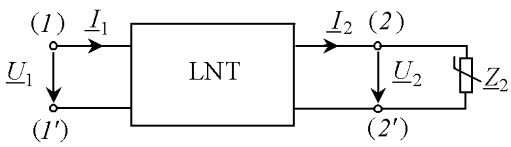

Fig. 1

As it is known (Philippow, 1963), if a nonlinear inertial element is excited with a harmonic signal, the response signal is harmonic too, so that the element's steady-state is a harmonic one, being possible, in this case, to utilize, in an advantageous manner, the symbolic proceeding utilizing complex quantities.

2. Utilized Method

The nonlinear inertial character of the complex impedance, $\underline{Z}_2$, can be render evident considering both this impedance and hers components, R_2 and X_2, as being functions of the amplitude, I_m, of an arbitrary harmonic current (I_m), as was considered in a previous paper (Rosman, 2005). Consequently relation (4) may be written

$$\underline{Z}_2(I_m)=R_2(I_m)+\mathrm{j}X_2(I_m). \tag{5}$$

Simultaneously with complex impedance, $\underline{Z}_2$,the signals on the two-port's gates become also functions of amplitude I_m, namely $\underline{I}_1(I_m)$, $\underline{U}_2(I_m)$, $\underline{I}_2(I_m)$, so that the transfer complex impedance (3) may be written

$$\underline{Z}_m(I_m) = \frac{\underline{U}_2(I_m)}{\underline{I}_1(I_m)} = R_m(I_m) + \mathrm{j}X_m(I_m). \tag{6}$$

Having in view two-port's eqs. (1) expression (6) becomes

$$\underline{Z}_m(I_m) = \frac{\underline{Z}_2(I_m)}{\underline{A}_{21}\underline{Z}_2(I_m) + \underline{A}_{22}}. \tag{7}$$

Evidently it is possible to consider, for simplicity, that $I_m = I_{2m}$.

In what follows the possibility that, in certain conditions, the modulus of complex transfer impedance, Z_{2m} , have extreme values, is studied. Similar studies concerning tha functions $\underline{k}_U(I_m)$ and $\underline{k}_I(I_m)$ moduli were performed in previous papers (Rosman, 2006, 2012), where $\underline{k}_U$ is the voltage transfer coefficient and $\underline{k}_I$ – the current transfer coefficient.

3. Differential Equation Satisfied by Function $X_2(R_2)$ (or $R_2(X_2)$)

Beforhand it is necessary to have in view that unlike the case when the LNT's receiver is linear, the complex transfer impedance being a function of two independent variables, R_2 and X_2 , when the LNT's receiver is non-linear inertial, this modulus is a function of a single independent variable, namely I_m .

If notations

$$x = R_2(I_m), \;\; y = X_2(I_m) \tag{8}$$

are introduced and taking into account relation (5), expression (7) becomes

$$\underline{Z}_m(I_m) = \frac{x(I_m) + \mathrm{j}y(I_m)}{\underline{A}_{21}\left[x(I_m) + \mathrm{j}y(I_m)\right] + \underline{A}_{12}}. \tag{9}$$

The complex transfer impedance's modulus is

$$Z_m(I_m) = \sqrt{\frac{x^2 + y^2}{A_{21}^2\left(x^2 + y^2\right) + 2\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)x - 2\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)y + A_{22}^2}}. \tag{10}$$

Having in view that x and y are functions of I_m it results that the derivative of expression (10) with respect to I_m is

$$\frac{\mathrm{d}Z_m}{\mathrm{d}I_m}=\left(x^2+y^2\right)^{-1/2}\left[A_{21}^2\left(x^2+y^2\right)+2\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)x-2\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)y+A_{22}^2\right]^{-3/2}\times$$
$$\times\left\{\left[2\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)\left(x^2-y^2\right)-2\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)xy+A_{22}^2x\right]\frac{\mathrm{d}x}{\mathrm{d}I_m}+\right. \tag{11}$$
$$\left.+\left[2\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)\left(x^2-y^2\right)+2\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)xy+A_{22}^2y\right]\frac{\mathrm{d}y}{\mathrm{d}I_m}\right\}.$$

Annulling this derivative it results the following differential eq.:

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)\left(x^2+y^2\right)-2\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)xy+A_{22}^2x}{\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)\left(x^2-y^2\right)+2\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)xy+A_{22}^2y}=0. \tag{12}$$

Using the notations

$$\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)=\alpha,\ \Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)=\beta,\ A_{22}^2=\gamma \tag{13}$$

the differential eq. (12) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\alpha\left(x^2+y^2\right)-2\beta xy+\gamma x}{\beta\left(x^2-y^2\right)+2\alpha xy+\gamma y}=0. \tag{14}$$

4. Integration of Differential Equation (14)

Differential eq. (14), of first order, belongs to the type

$$P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=0, \tag{15}$$

where

$$P(x,y)=\alpha\left(x^2+y^2\right)-2\beta xy+\gamma x,\ Q(x,y)=\beta\left(x^2-y^2\right)+2\alpha xy+\gamma y. \tag{16}$$

Accordingly

$$\frac{\partial P}{\partial y}=-2(\beta x+\alpha y),\quad \frac{\partial Q}{\partial x}=2(\beta x+\alpha y), \tag{17}$$

so $\partial P/\partial y\neq\partial Q/\partial x$ and, consequently, expression (15) not represents an exact total differential. It results that isn't possible to integrate the differential eq. (14), in general case, than with numerical methods.

4.1. Particular Cases

In the particular cases when either α = 0, or β = 0, the differential eq. (14) becomes more simple, here integration being possible using analytical proceedings.

a) *Case* α =0 $\left(\Re e\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)=0\right)$. This particular situation takes place when P_{20} = 0; representing the LNT through the equivalent scheme in T as in Fig. 2, the case α = 0 is realized when the complex impedances $\underline{\zeta}_2$ and $\underline{\zeta}_3$ are pure reactive. In this case differential eq. (14) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x\left(2\beta y-\gamma\right)}{\beta\left(x^2-y^2\right)+\gamma y}. \tag{18}$$

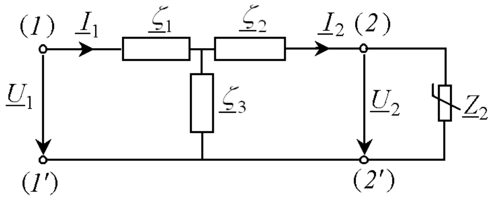

Fig. 2

Performing the dependent variable changing

$$2\beta y-\gamma=z \tag{19}$$

differential eq. (18) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{8\beta^2 xz}{4\beta^2x^2-z^2+\gamma^2}. \tag{20}$$

The integration of this eq. is simpler when this one is written as

$$\frac{\mathrm{d}x}{\mathrm{d}z}=\frac{4\beta^2x^2-z^2+\gamma^2}{8\beta^2xz}=\frac{1}{2}\cdot\frac{x}{z}-\frac{z^2-\gamma^2}{8\beta^2xz}, \tag{21}$$

belonging to Bernoulli type (Corduneanu, 1981), having the general form

$$\frac{\mathrm{d}x}{\mathrm{d}z}=m(z)x+n(z)x^p,\ x\in\Im, \tag{22}$$

with

$$m(z)=\frac{1}{2z},\ n(z)=\frac{z^2-\gamma^2}{8b^2z},\ p=-1, \tag{23}$$

the functions $m(z)$, $n(z)$ being continuous in range $\Im$.

In view to integrate differential eq. (21), this one is divided by *x* and the dependent variable changing

$$x^2 = \lambda \tag{24}$$

is performed so that eq. (21) becomes

$$\frac{\mathrm{d}\lambda}{\mathrm{dz}} = \frac{\lambda}{z} - \frac{z^2 - \gamma^2}{4\beta^2 z}, \tag{25}$$

that is a linear differential eq. having the form

$$\frac{\mathrm{d}\lambda}{\mathrm{dz}} + M(z)\lambda + N(z) = 0 \tag{26}$$

having the solution (Corduneanu, 1981)

$$\lambda(z) = \mathrm{e}^{-\int M(z)\mathrm{dz}}\left[C - \int N(z)\mathrm{e}^{\int M(z)\mathrm{dz}}\mathrm{dz}\right], \tag{27}$$

where

$$M(z) = \frac{\lambda}{z},\ \ N(z) = \frac{z^2 - \gamma^2}{4\beta^2 z} \tag{28}$$

and C is an integration constant.

Differential eq.'s solution (27) can be written as

$$\lambda(z) = Cz - \frac{1}{4\beta^2}\left(z^2 + \gamma^2\right). \tag{29}$$

Having in view relations (19) and (24) expression (29) becomes

$$x = \sqrt{C(2\beta y - \gamma) - \frac{1}{4\beta^2}\left[(2\beta y - \gamma)^2 + \gamma^2\right]}. \tag{30}$$

Since *x* represents a resistance (see rel. (8)) it is necessary that the inequality

$$C(2\beta y - \gamma) - \frac{1}{4\beta^2}\left[(2\beta y - \gamma)^2 + \gamma^2\right] \geq 0 \tag{31}$$

be satisfied, which may be written as

$$y^2 - \left(2\beta C + \frac{\gamma}{\beta}\right) - \gamma C + \frac{\gamma^2}{2\beta^2} \leq 0. \tag{32}$$

The trinomial's roots from the right side of inequality(32) are

$$y', y'' = \beta C + \frac{\gamma}{2\beta} \pm \frac{1}{2}\sqrt{2\beta^2 C^2 - \frac{\gamma^2}{\beta^2}}. \tag{33}$$

As regards the integration constant, C, this one must satisfy the supplementary inequality

$$C \geq \frac{\gamma}{2\beta^2} > 0, \tag{34}$$

where relation (13_3) was taken into accont.

It is advantageous to write relation (30) as

$$x^2 + y^2 - \left(2\beta C + \frac{\gamma}{\beta}\right)y + \frac{\gamma^2}{2\beta^2} + C\gamma \geq 0, \tag{35}$$

the equality with zero representing the eq. of a circle (Fig. 3) having the center

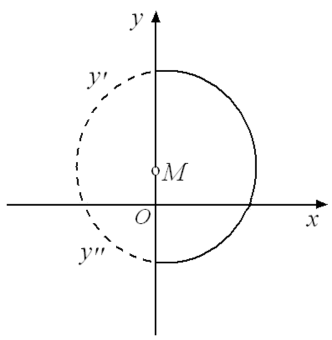

Fig. 3

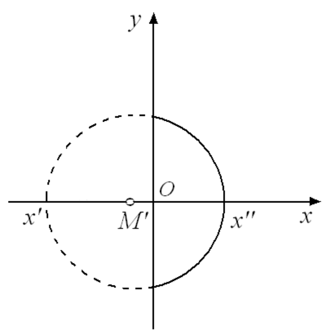

Fig. 4

in $M(0, \beta C + \gamma / 2\beta)$ and the radius $\sqrt{\beta^2 C^2 + \gamma^2 / 4\beta^2}$. Having in view relation (13) it results that the circle's center, M, can be situated both on semi-axis $y > 0$ or $y < 0$. In the same time, taking into account the notations (8) too, circle's (35) eq. may be written as

$$R_2^2(I_m) + X_2^2(I_m) - \left[2C\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right) + \frac{A_{22}^2}{\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)}\right]X_2(I_m) + \\ + \frac{A_{22}^4}{2\left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]^2} + CA_{22}^2 = 0, \tag{36}$$

having the center in $M\left[0, C\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right) - A_{22}^{2} \big/ 2\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)\right]$ and the radius $\sqrt{C^{2}\left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)\right]^{2} - A_{22}^{2} \big/ 4\left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)\right]^{2}}$. At the same time relations (33) and (34) become

$$y_2^{'}, y_2^{''} = C\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right) + \frac{A_{22}^{2}}{2\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)} \pm \\ \pm\sqrt{4C^{2}\left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)\right]^{2} - A_{22}^{4} \big/ \left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)\right]^{2}}, \tag{37}$$

respectively

$$C \geq \frac{A_{22}^{2}}{2\left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right)\right]^{2}} > 0. \tag{38}$$

Since x and y have the significances of a resistance, respectively of a reactance, it is evidently that only the half circle situated in the semi-plane $x > 0$ has a physical meaning. This half of circle may be considered as representing the geometric-locus diagram of the complex impedance $\underline{Z}_2(I_m)$ corresponding to the extreme values of complex transfer impedance, $\underline{Z}_m$, when the LNT's fundamental parameters satisfy relation $\Re e\left(\underline{A}_{21}\underline{A}_{22}^{*}\right) = 0$.

b) *Case* $\beta = 0$ $\left(\Im m\left(\underline{A}_{21}\underline{A}_{22}^{*}\right) = 0\right)$. This situation is realized when Q_{20} = = 0. Considering the LNT's equivalent scheme in T represented in Fig. 2, this case corresponds to the situation when complex impedance $\underline{\zeta}_2$ and $\underline{\zeta}_3$ have equal and of opposite sign reactances $\left(\Im m\left(\underline{\zeta}_2\right) = -\Im m\left(\underline{\zeta}_3\right)\right)$. In this particular case the differential eq. (14) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{y(2\alpha x + \gamma)}{\alpha\left(x^{2} - y^{2}\right) + \gamma x} = 0, \tag{39}$$

similar to the differential eq. (18). Consequently, using an analogous proceeding as in the previous particular case, is possible to integrate differential eq. (39) obtaining

$$y = \sqrt{C'(2\alpha x + \gamma) - \frac{1}{4\alpha^{2}}\left[(2\alpha x + \gamma)^{2} + \gamma^{2}\right]}, \tag{40}$$

where C' is an integration constant. Because y represents a reactance (see rel. (8)) it must be a real quantity so that the inequality

$$C'(2\alpha x+\gamma)-\frac{1}{4\alpha^2}\left[(2\alpha x+\gamma)^2+\gamma^2\right]\geq 0 \tag{41}$$

must be fulfilled, which is equivalent with the inequality

$$x^2-\left(2\alpha C'-\frac{\gamma}{\alpha}\right)x-\gamma C'+\frac{\gamma^2}{2\alpha^2}\leq 0\,. \tag{42}$$

The roots of the trinomial from the left side of inequality (42) are

$$x',x''=\alpha C'-\frac{\gamma}{2\alpha}\pm\frac{1}{2}\sqrt{2\alpha^2C'^2-\frac{\gamma^2}{\alpha^2}}\,. \tag{43}$$

The integration constant, C', satisfies the inequality

$$C'\geq\frac{\gamma}{2\alpha^2}>0, \tag{44}$$

where notations (13) were taken into account.

Relation (40) can be written, more advantageously, as

$$x^2+y^2-\left(\frac{\gamma}{\alpha}-2\alpha C'\right)x+\frac{\gamma^2}{2\alpha^2}-\gamma C'=0, \tag{45}$$

which represents the eq. of a circle (Fig. 4) having the center in $M'(\gamma/2\alpha-\alpha C',0)$ and the radius $\sqrt{\alpha^2C'^2-\gamma^2/4\alpha^2}$. Having in view inequalities (43) it results that the circle's center, M', is situated constantly on the axis $x < 0$.

In this case too, having in view that x and y have the significances of a resistance, respectively of a reactance, only the circle's arc situated in the half-plane $x \geq 0$ have a physical meaning. This circle's arc constitutes, properly, the complex impedance's $\underline{Z}_2(I_m)$ geometric-locus diagram wich corresponds to the extreme values of transfer complex impedance's modulus, Z_m , when the LNT's fundamental parameters satisfy relation $\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)=0$.

Having in view the relations (8) and (13) significations eq. (45) may be written as

$$\begin{gathered}R_2^2(I_m)+X_2^2(I_m)-\left[\frac{A_{22}^2}{\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)}-2C'\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]R_2(I_m)+\\+\frac{A_{22}^4}{2\left[\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]^2}-C'A_{22}^2=0,\end{gathered} \tag{46}$$

having the center in $M'\left[A_{22}^2\big/2\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)-C'\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right),0\right]$ and the radius $\sqrt{C'^2\left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]^2-A_{22}^4\big/\left[\Im m\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]^2}$. In the same time relations (43) and (44) become

$$R_2'(I_m),R_2''(I_m)=C'\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)-\frac{A_{22}^2}{2\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)}\pm$$
$$\pm\frac{1}{2}\sqrt{4C'^2\left[\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]^2-A_{22}^4\big/\left[\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]^2}, \tag{47}$$

respectively

$$C'\geq\frac{A_{22}^2}{2\left[\Re e\left(\underline{A}_{21}\underline{A}_{22}^*\right)\right]^2}>0. \tag{38}$$

5. Conclusions

1. The differential equation satisfied by function $X_2(R_2)$ is established, where $\underline{Z}_2 = R_2 + jX_2$ is the nonlinear, inertial receiver's complex impedance of a linear and non-autonomous two-port (in a restreint sense), in harmonic steady-state, so that the two-port's transfer complex impedance's modulus have extreme values

2. The established differential equation, which is nonlinear of first order, is integrated analytically in two particular cases. In each case the equation $X_2(R_2)$ represents a circle which is, in the half-plane $R_2 \geq 0$, the geometric-locus diagram of the receiver's complex impedance, $\underline{Z}_2$, corresponding to the extreme values of the two-port's transfer complex impedance, $\underline{Z}_m$, for different values of the secondary current amplitude (and, implicitely, of the primary voltage amplitude).

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IMPEDANŢA DE TRANSFER A UNUI CUADRIPOL LINIAR CU RECEPTOR NELINIAR

(Rezumat)

Se stabileşte ecuaţia diferenţială, neliniară, de primul ordin, satisfăcută de funcţia $X_2(R_2)$, unde $\underline{Z}_2 = R_2 + jX_2$ este impedanţa complexă a receptorului neliniar, inerţial şi pasiv, a unui cuadripol (în sens restrâns) liniar şi neautonom, în regim permanent armonic, astfel încât modulul impedanţei de transfer a cuadripolului să aibă o valoare extremă.

Ecuaţia diferenţială stabilită se integrează în două cazuri particulare, în care aceasta este de tip Bernoulli, permiţând obţinerea unei soluţii analitice.