BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Tomul LIX (LXIII), Fasc. 2, 2013
Secţia
ELECTROTEHNICĂ. ENERGETICĂ. ELECTRONICĂ

SKIN EFFECT IN A CONDUCTING PLATE WITH RECTANGULAR SECTION, IN PERIODICAL NON-HARMONIC STEADY-STATE

BY

HUGO ROSMAN*

"Gheorghe Asachi" Technical University of Iaşi
Faculty of Electrical Engineering

Received: May 9, 2013
Accepted for publication: June 28, 2013

Abstract. Using the symbolic method based on representation of periodical non-harmonic signals through hypercomplex "images", the hypercomplex moduli of electromagnetic state vectors $\hat{\mathbf{E}}_{\text{int}}$ and $\hat{\mathbf{H}}_{\text{int}}$ are determined in a popint situated inside of a conducting plate with rectangular section through which flows a periodical non-harmonic current. In the same time the hypercomplex modulus of Poynting's vector in a point situated on the conducting plate's surface is determined too, permitting to perform the calculus of active and reactive powers on plate's length unit and width unit. It was taken into account that the inferior order current harmonics produce a weak skin effect, those of superior order generating a net one.

Key words: conducting plate; periodical non-harmonic steady-state; hypercomplex symbolic method; hypercomplex moduli of vectors $\hat{\mathbf{E}}_{\text{int}}$, $\hat{\mathbf{H}}_{\text{int}}$, $\hat{\mathbf{S}}$; active and reactive powers.

1. Introduction

Let be a conducting plate having a rectangular section, with the width *d*, through which flows a harmonic current, $i_l = \sqrt{2} I_l \sin \omega t$, on the length's unit of

* *e-mail*: adi_rotaru2005@yahoo.com

this one. The plate is refered to a triorthogonal system of co-ordinates, the plate's symmetry plane being *xOz*, and the *Oy*-axis, normal on the plate's faces (Fig. 1). In this case the field's electromagnetic state vectors in a point situated inside of the plate have the complex effective values (Mocanu, 1984)

$$\mathbf{E}_{\text{int}}(x) = \mathbf{j}\frac{\underline{\gamma}\underline{I}_l}{2\sigma}\cdot\frac{\text{ch}\underline{\gamma}x}{\text{sh}\dfrac{\underline{\gamma}d}{2}},\quad \mathbf{H}_{\text{int}}(x) = \mathbf{k}\frac{\underline{I}_l}{2\sigma}\cdot\frac{\text{sh}\underline{\gamma}x}{\text{sh}\dfrac{\underline{\gamma}d}{2}},\quad x\in\left[0,\frac{d}{2}\right], \tag{1}$$

in harmonic steady-state, where $\underline{I}_l$ is the complex effective value of the current i_l.

Consequently the complex modulus of Poynting's vector is

$$\underline{S}\left(\frac{d}{2}\right) = \frac{\underline{\gamma}I_l^2}{4\sigma}\cdot\frac{\text{ch}\dfrac{\underline{\gamma}d}{2}}{\text{sh}\dfrac{\underline{\gamma}d}{2}} = \frac{\alpha I_l^2}{4\sigma}\cdot\frac{\text{ch}\alpha d+\sin\alpha d+\text{j}\left(\text{ch}\alpha d-\sin\alpha d\right)}{\text{ch}\alpha d-\cos\alpha d}, \tag{2}$$

in a point situated on the plate's symmetry axis ($x = d/2$). Here

$$\underline{\gamma} = (1+\text{j})\alpha,\quad \alpha = \sqrt{\frac{\omega\mu\sigma}{2}}, \tag{3}$$

represent the complex propagation constant and, respectively, the attenuation constant; σ and μ are the conductor's material constants.

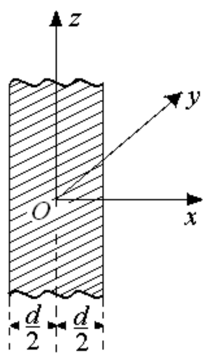

Fig. 1 – Transversal section through the conducting plate.

Utilizing the well known series developments

$$\begin{aligned}&\sin m = m-\frac{m^3}{6}+\frac{m^5}{120}-...,\ \text{sh}\ m = m+\frac{m^3}{6}+\frac{m^5}{120}+...,\\&\cos m = 1-\frac{m^2}{2}+\frac{m^4}{24}-...,\ \text{ch}\ m = 1+\frac{m^2}{2}+\frac{m^4}{24}+...,\end{aligned} \tag{4}$$

with $m=\underline{\gamma}x,\ \underline{\gamma}d/2,\ \alpha d,$ if from these series are retained only the first two terms relations (1) lead to

$$\underline{E}_{\text{int}}(x) = \frac{12\underline{I}_l\left(2+\underline{\gamma}^2x^2\right)}{d\sigma\left(24+\underline{\gamma}^2d^2\right)},\ \underline{H}_{\text{int}}(x) = \frac{4x\underline{I}_l\left(6+\underline{\gamma}^2x^2\right)}{d\left(24+\underline{\gamma}^2d^2\right)}, \tag{5}$$

which characterize the *weak skin effect* ($\alpha d << 1$).

On the contrary, in case of the *net skin effect* the asymptotic values of hyperbolic trigonometric functions being

$$\mathrm{ch}\, m \approx \mathrm{e}^{m}, \quad \mathrm{sh}\, m \approx \mathrm{e}^{m}, \tag{6}$$

expressions (1) permit to consider that

$$\underline{E}_{\mathrm{int}}(x) = \frac{\gamma \underline{I}_l}{2\sigma} \mathrm{e}^{-\underline{\gamma}\left(\frac{d}{2}-x\right)}, \quad \underline{H}_{\mathrm{int}}(x) = \frac{\underline{I}_l}{2} \mathrm{e}^{-\underline{\gamma}\left(\frac{d}{2}-x\right)}. \tag{7}$$

The aim of this work is to study the skin effect in a conducting plate having a rectangular section, in *periodical non-harmonic state.*

In what follows the dispersion *vs*. time, of material constants, σ, μ, is neglected.

2. The Study Method

With the view of the skin effect study in periodical non-harmonic steady-state is advantageously to utilize the symbolic method which permits the representation of periodical non-harmonic signals through hypercomplex "images". Elaborated by B.A. Rozenfeld, (1949), and completed by the author (Rosman, 2010), this method consists in attaching to a periodical non-harmonic signal developed in Fourier series

$$a(t) = \sum_{k=0}^{\infty} A_k^{'} \cos k\omega t + \sum_{k=0}^{\infty} A_k^{''} \sin k\omega t\,, \tag{8}$$

considered as "cartesian" form, or

$$a(t) = \sqrt{2}\sum_{k=0}^{\infty} A_k \sin\left(k\omega t + \alpha_k\right), \tag{9}$$

named "polar" form, with

$$A_k = \frac{1}{\sqrt{2}}\sqrt{A_k^{'2} + A_k^{''2}}, \quad \alpha_k = \arctan\frac{A_k^{''}}{A_k^{'}}, \tag{10}$$

the hypercomplex "image"

$$\hat{A} = \sum_{k=0}^{\infty} 1_k A_k^{'} + \sum_{k=0}^{\infty} \mathrm{j}_k A_k^{''}, \tag{11}$$

in the variant proposed by B.A. Rozenfeld (*op. cit.*), or

$$\hat{a} = \sum_{k=0}^{\infty} 1_k A_k \cos\left(k\omega t + \alpha_k\right) + \sum_{k=0}^{\infty} \mathrm{j}_k A_k \sin\left(k\omega t + \alpha_k\right), \tag{12}$$

in that conceived by the author. Here functions 1_k , j_k are orthonormalized, the defined algebra in this way being commutative, representing a direct sum of real numbers field (generated by 1_0) and the numberable set of complex numbers set (generated by the pair 1_k , j_k). the unit element of the so defined algebra is

$$\sum_{k=0}^{\infty} 1_k = 1_0. \tag{13}$$

In the same time relations

$$1_k^2 = 1_k,\ \mathrm{j}_k^2 = -1_k,\ 1_k \mathrm{j}_k = \mathrm{j}_k 1_k = \mathrm{j}_k,\ 1_p 1_q = 1_q 1_p = 1_p \mathrm{j}_q = \mathrm{j}_q 1_p = \mathrm{j}_p \mathrm{j}_q = \mathrm{j}_q \mathrm{j}_p = 0,\ \left(p \neq q\right), \tag{14}$$

are satisfied.

It is necessary to underline that the symbolic identity

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} = \sum_{k=0}^{\infty} \left(\mathrm{j}_k k\omega\right)^m,\ m \in \mathbb{N}, \tag{15}$$

renders evident the advantage of this method in the sense that it permits the “algebrization” of differential operations with respect to time. Consequently the integro-differential eqs. satisfied by the signals become algebraic ones having as unknowns the signal’s hypercomplex “images”. The advantage is similar to that conceded by the symbolic complex method in case of harmonic signals.

3. Hypercomplex Vectors $\hat{\mathbf{E}}_{\mathrm{int}}(x)$, $\hat{\mathbf{H}}_{\mathrm{int}}(x)$, $\hat{\mathbf{S}}(d/2)$

If $\hat{\mathbf{E}}_{\mathrm{int}}(x)$, $\hat{\mathbf{H}}_{\mathrm{int}}(x)$ represent the hypercomplex state vectors of an electromagnetic field, in periodical non-harmonic steady-state, in a point situated inside of a conducting plate having a rectangular section, through which flows the current

$$i_l = \sqrt{2} \sum_{k=0}^{\infty} I_{l_k} \sin\left(k\omega t + \gamma_{i_k}\right), \tag{16}$$

on the length’s unit of the plate, theirs expressions are similar to (1), substituting in these ones $\underline{I}_{\mathrm{l}}$ with

$$\hat{I}_l = \sum_{k=0}^{\infty}\left(1_k I'_{l_k} + \mathrm{j}_k I''_{l_k}\right) \tag{17}$$

and $\underline{\gamma}$ with (Rosman, 1979)

$$\hat{\gamma} = \sum_{k=0}^{\infty}\left(1_k + \mathrm{j}_k\right)\alpha_k = \sqrt{\sum_{k=0}^{\infty}\mathrm{j}_k k\omega\sigma\mu}, \tag{18}$$

where $\hat{\gamma}$ represents the hypercomplex propagation constant and

$$\alpha_k = \sqrt{\frac{k\omega\sigma\mu}{2}} \tag{19}$$

– the attenuation constant of *k*-th order harmonic of the electromagnetic waves inside of the plate. It results

$$\hat{E}_{\mathrm{int}}(x) = \frac{\hat{\gamma}\hat{I}_l}{2\sigma}\cdot\frac{\mathrm{ch}\hat{\gamma}x}{\mathrm{sh}\frac{\hat{\gamma}d}{2}}, \quad \hat{H}_{\mathrm{int}}(x) = \frac{\hat{I}_l}{2}\cdot\frac{\mathrm{sh}\hat{\gamma}x}{\mathrm{sh}\frac{\hat{\gamma}d}{2}}. \tag{20}$$

Knowing expressions (20) of hypercomplex moduli of $\hat{\mathbf{E}}_{\mathrm{int}}(x)$ and $\hat{\mathbf{H}}_{\mathrm{int}}(x)$ vectors it is possible to determine the Poynting's hypercomplex vector on the plate's length's and width's unit

$$\hat{\mathbf{S}}_l(x) = \hat{\mathbf{E}}_{\mathrm{int}}(x)\times\hat{\mathbf{H}}^*_{\mathrm{int}}(x), \tag{21}$$

having the hypercomplex modulus, in $x = d/2$,

$$\hat{S}_l\left(\frac{d}{2}\right) = \hat{E}\left(\frac{d}{2}\right)\hat{H}^*\left(\frac{d}{2}\right), \tag{22}$$

where

$$\hat{E}\left(\frac{d}{2}\right) = \frac{\hat{\gamma}\hat{I}_l}{2\sigma}\cdot\frac{\mathrm{ch}\frac{\hat{\gamma}d}{2}}{\mathrm{sh}\frac{\hat{\gamma}d}{2}}, \quad \hat{H}\left(\frac{d}{2}\right) = \frac{\hat{I}_l}{2}. \tag{23}$$

In expression (22) $\hat{H}^*(d/2)$ represents the hypercomplex conjugate of $\hat{H}(d/2)$, namely $\hat{H}^*(d/2) = \hat{I}^*_l/2$, with

$$\hat{I}_l^* = \sum_{k=0}^{\infty}\left(1_k I'_{l_k} - \mathrm{j}_k I''_{l_k}\right) \tag{24}$$

so that $\hat{I}_l \hat{I}_l^* = I_l^2 = \sum_{k=0}^{\infty}\left(I'^2_{l_k} + I''^2_{l_k}\right)$. Having in view relations (22),…,(24) it results

$$\hat{S}_l\left(\frac{d}{2}\right) = \frac{\hat{\gamma}\hat{I}_l^2}{4\sigma}\cdot\frac{\mathrm{ch}\,\frac{\hat{\gamma}d}{2}}{\mathrm{sh}\,\frac{\hat{\gamma}d}{2}}. \tag{25}$$

In a previous work (Rosman, 1960) it was established that in periodical non-harmonic steady-state a *hypercomplex apparent power* may be defined namely

$$\hat{s} = \sum_{k=0}^{\infty} 1_k P_k + \sum_{k=0}^{\infty} \mathrm{j}_k Q_k, \tag{26}$$

where

$$P = \sum_{k=0}^{\infty} P_k, \quad Q = \sum_{k=0}^{\infty} Q_k \tag{27}$$

represent the active, respectively the reactive power. Considering all the powers (active, reactive and hypercomplex apparent) refered to plate's length's unit and width's unit, then

$$\hat{s}_l = \hat{S}_l\left(\frac{d}{2}\right) = \sum_{k=0}^{\infty} 1_k P_{l_k} + \sum_{k=0}^{\infty} \mathrm{j}_k Q_{l_k}. \tag{28}$$

In view to obtain the expressions of active power, P_l , and reactive power, Q_l , it is necessary to amplify the fraction $\mathrm{ch}(\hat{\gamma}d/2)/\mathrm{sh}(\hat{\gamma}d/2)$, with the hypercomplex conjugate of his denominator, $\mathrm{sh}\left(\hat{\gamma}^* d/2\right)$. Having in view (18) and performing the calculus it results

$$\frac{\mathrm{ch}\,\frac{\hat{\gamma}d}{2}}{\mathrm{sh}\,\frac{\hat{\gamma}d}{2}} = \frac{\mathrm{sh}\left(\sum_{k=0}^{\infty} 1_k \alpha_k d\right) - \mathrm{sh}\left(\sum_{k=0}^{\infty} \mathrm{j}_k \alpha_k d\right)}{\mathrm{ch}\left(\sum_{k=0}^{\infty} 1_k \alpha_k d\right) - \mathrm{ch}\left(\sum_{k=0}^{\infty} \mathrm{j}_k \alpha_k d\right)}. \tag{29}$$

Amplifying again the right side of (29) with the hypercomplex conjugate of the denominator, the expression

$$\frac{\operatorname{ch}\frac{\hat{\gamma}d}{2}}{\operatorname{sh}\frac{\hat{\gamma}d}{2}}=\frac{\sum_{k=0}^{\infty}1_k\operatorname{sh}\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\operatorname{ch}\left(\alpha_l d\right)-\sum_{k=0}^{\infty}\mathrm{j}_k\sin\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\cos\left(\alpha_l d\right)}{\prod_{l=0}^{\infty}\operatorname{ch}\left(\alpha_l d\right)-\prod_{l=0}^{\infty}\cos\left(\alpha_l d\right)} \tag{30}$$

is obtained.

Having in view (18) and taking into account relations (25),…,(28) it results

$$\hat{s}_l=\sum_{k=0}^{\infty}1_k P_{l_k}+\sum_{k=0}^{\infty}\mathrm{j}_k Q_{l_k}=\frac{I_l^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=0}^{\infty}\left(1_k+\mathrm{j}_k\right)\sqrt{k}\times$$
$$\times\frac{\sum_{k=0}^{\infty}1_k\operatorname{sh}\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\operatorname{ch}\left(\alpha_l d\right)-\sum_{k=0}^{\infty}\mathrm{j}_k\sin\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\cos\left(\alpha_l d\right)}{\prod_{l=0}^{\infty}\operatorname{ch}\left(\alpha_l d\right)-\prod_{l=0}^{\infty}\cos\left(\alpha_l d\right)}, \tag{31}$$

where relations

$$\operatorname{ch}\left(1_k\lambda\right)=\operatorname{ch}\lambda,\quad \operatorname{sh}\left(1_k\lambda\right)=1_k\operatorname{sh}\lambda,$$
$$\operatorname{ch}\left(\mathrm{j}_k\lambda\right)=\cos\lambda,\quad \operatorname{sh}\left(\mathrm{j}_k\lambda\right)=\mathrm{j}_k\operatorname{sh}\lambda, \tag{32}$$

and (14) were taken into account too.

From relation (31) it results

$$P_l=\frac{I_l^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\cdot\frac{\sum_{k=0}^{\infty}\sqrt{k}\,\operatorname{sh}\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\operatorname{ch}\left(\alpha_l d\right)+\sum_{k=0}^{\infty}\sqrt{k}\sin\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\cos\left(\alpha_l d\right)}{\prod_{l=0}^{\infty}\operatorname{ch}\left(\alpha_l d\right)-\prod_{l=0}^{\infty}\cos\left(\alpha_l d\right)},$$
$$Q_l=\frac{I_l^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\cdot\frac{\sum_{k=0}^{\infty}\sqrt{k}\,\operatorname{sh}\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\operatorname{ch}\left(\alpha_l d\right)-\sum_{k=0}^{\infty}\sqrt{k}\sin\left(\alpha_k d\right)\prod_{\substack{l=0\\l\neq k}}^{\infty}\cos\left(\alpha_l d\right)}{\prod_{l=0}^{\infty}\operatorname{ch}\left(\alpha_l d\right)-\prod_{l=0}^{\infty}\cos\left(\alpha_l d\right)}. \tag{33}$$

In general case, when fundamental term's frequency of the current, i_l, is enough low, in frequency spectrum of this one it is possible to define three domains where the skin effect produced by the current harmonics is qualitatively different namely: a) the domain [*f*, *pf*], in which the frequencies are enough low, so that the skin effect may be considered *weak*; b) the domain [*pf*, *qf*], in which the skin effect is *medium*; c) the domain [*qf*, ∞), in which the skin effect is *net*. Here *f* represents the fundamental term's frequency of the current, $p, q \in \Im$, with $1 < p < q$.

3.1. Skin Effect Produced by Low Harmonics

Taking into account that the skin effect produced by the current's harmonics with the frequencies situated in the domain [*f*, *pf*] is weak, the state vector's hypercomplex moduli may be calculated using the approximate relations (5), where $\hat{I}_l$, given by expression (17), is substituted with

$$\hat{I}_{l_p} = \sum_{k=0}^{p}\left(1_k I'_{l_k} + \mathrm{j}_k I''_{l_k}\right), \tag{34}$$

and the hypercomplex constant of propagation, $\hat{\gamma}$, with

$$\hat{\gamma}_p = \sum_{k=0}^{p}\left(1_k + \mathrm{j}_k\right)\alpha_k = \sqrt{\sum_{k=0}^{p} \mathrm{j}_k k\omega\mu\sigma}. \tag{35}$$

It results

$$\begin{aligned} \hat{E}_{\mathrm{int}\,p}(x) &= \frac{12\hat{I}_{l_p}}{d\sigma}\cdot\frac{\sum_{k=0}^{p}\left(1_k 12 + \mathrm{j}_k k\omega\sigma\mu x^2\right)}{\sum_{k=0}^{p}\left(1_k 24 + \mathrm{j}_k k\omega\sigma\mu d^2\right)}, \\ \hat{H}_{\mathrm{int}\,p}(x) &= \frac{4x\hat{I}_{l_p}}{d}\cdot\frac{\sum_{k=0}^{p}\left(1_k 6 + \mathrm{j}_k k\omega\sigma\mu x^2\right)}{\sum_{k=0}^{p}\left(1_k 24 + \mathrm{j}_k k\omega\sigma\mu d^2\right)}. \end{aligned} \tag{36}$$

If relations (36) are amplified with $\sum_{k=0}^{p}\left(1_k 24 - \mathrm{j}_k k\omega\sigma\mu d^2\right)$, representing the hypercomplex conjugate of the denominator (the same) of expressions (36) and having in view relations (14), it results

$$\hat{E}_{\mathrm{int}\,p}(x)=\frac{12\hat{I}_{l_p}}{d\sigma}\cdot\frac{\sum\limits_{k=0}^{p}\left[1_k\left(48+k^2\omega^2\sigma^2\mu^2d^2x^2\right)+\mathrm{j}_k\,2k\omega\sigma\mu\left(12x^2-d^2\right)\right]}{576+\omega^2\sigma^2\mu^2d^4\sum\limits_{k=0}^{p}k^2},$$
$$\hat{H}_{\mathrm{int}\,p}(x)=\frac{4x\hat{I}_{l_p}}{d}\cdot\frac{\sum\limits_{k=0}^{p}\left[1_k\left(144+k^2\omega^2\sigma^2\mu^2d^2x^2\right)-\mathrm{j}_k\,6k\omega\sigma\mu\left(d^2-4x^2\right)\right]}{576+\omega^2\sigma^2\mu^2d^4\sum\limits_{k=0}^{p}k^2}. \quad (37)$$

In the same time the hypercomplex Poynting vector's modulus, taking in account the series developments (4) and retaining, in this case too, only two terms of these series, is

$$\hat{S}_{l_p}\left(\frac{d}{2}\right)=\frac{3I_{l_p}^2}{2\sigma d}\cdot\frac{8+\hat{\gamma}_p^2d^2}{24+\hat{\gamma}_p^2d^2}. \quad (38)$$

with $\hat{\gamma}_p$ given by (35). Having in view (18), (19) and (13) relation (38) becomes

$$\hat{S}_{l_p}\left(\frac{d}{2}\right)=\frac{3I_{l_p}^2}{2\sigma d}\cdot\frac{\sum\limits_{k=0}^{p}\left(1_k8+\mathrm{j}_kk\omega\sigma\mu d^2\right)}{\sum\limits_{k=0}^{p}\left(1_k24+\mathrm{j}_kk\omega\sigma\mu d^2\right)}. \quad (39)$$

Amplifying the right side of relation (39) with denominator's hypercomplex conjugate and performing the calculus it results

$$\hat{S}_{l_p}\left(\frac{d}{2}\right)=\frac{3\hat{I}_{l_p}^2}{2\sigma d}\cdot\frac{\sum\limits_{k=0}^{p}\left[1_k\left(192+k^2\omega^2\sigma^2\mu^2d^4\right)+\mathrm{j}_k16k\omega\sigma\mu d^2\right]}{576+\omega^2\sigma^2\mu^2d^4\sum\limits_{k=0}^{p}k^2}. \quad (40)$$

On the basis of relations (26),...,(28) expression (40) permits to determine both the active power

$$P_{l_p}\left(\frac{d}{2}\right)=\frac{3\hat{I}_{l_p}^2}{2\sigma d}\cdot\frac{192+\omega^2\sigma^2\mu^2d^4\sum\limits_{k=0}^{p}k^2}{576+\omega^2\sigma^2\mu^2d^4\sum\limits_{k=0}^{p}k^2} \quad (41)$$

and the reactive power

$$Q_{l_p}\left(\frac{d}{2}\right)=\frac{3\hat{I}_{l_p}^2}{2\sigma}\cdot\frac{8\omega\sigma\mu d\sum_{k=0}^{p}k}{576+\omega^2\sigma^2\mu^2 d^4\sum_{k=0}^{p}k^2}, \tag{42}$$

reported to conducting plate's length unit and width unit. But (Ryžik & Gradshtein, 1951)

$$\sum_{k=0}^{p}k=\frac{p(p+1)}{2},\quad \sum_{k=0}^{p}k^2=\frac{p(p+1)(2p+1)}{6}, \tag{43}$$

so that expressions (41) and (42) become

$$P_{l_p}=\frac{3\hat{I}_{l_p}^2}{2\sigma d}\cdot\frac{1152+p(p+1)(2p+1)\omega^2\sigma^2\mu^2 d^4}{3456+p(p+1)(2p+1)\omega^2\sigma^2\mu^2 d^4}, \tag{44}$$

respectively

$$Q_{l_p}=\frac{72\hat{I}_{l_p}^2}{\sigma}\cdot\frac{p(p+1)\omega\sigma\mu d}{3456+p(p+1)(2p+1)\omega^2\sigma^2\mu^2 d^4}. \tag{45}$$

3.2. Skin Effect Produced by Medium Harmonics

In the current's, i_l , harmonics frequencies spectrum [*pf*, *qf*] the skin effect is a medium one, expressions (20), (25) allowing to perform the hypercomplex vectors moduli $\hat{E}_{\text{int}_{pq}}(x)$, $\hat{H}_{\text{int}_{pq}}(x)$ and, respectively, $\hat{S}_{l_{pq}}(d/2)$, with the specification that the hypercomplex "image" of the current, $\hat{I}_l$, must be substituted with

$$\hat{I}_{l_{pq}}=\sum_{k=p}^{q}\left(1_k I'_{l_k}+\mathrm{j}_k I''_{l_k}\right), \tag{46}$$

which corresponds to the current's harmonics having the frequencies comprised between *pf* and *qf*. In the same time the hypercomplex propagation constant, $\hat{\gamma}$ (s. rel. (18)), must be substituted with

$$\hat{\gamma}_{pq}=\sum_{k=p}^{q}(1_k+\mathrm{j}_k)\alpha_k=\sqrt{\sum_{k=p}^{q}\mathrm{j}_k k\omega\mu\sigma}. \tag{47}$$

The explicit determination, in this case, of hypercomplex vectors moduli $\hat{E}_{\mathrm{int}_{pq}}(x)$, $\hat{H}_{\mathrm{int}_{pq}}(x)$ is quite difficult.

As regards expressions of active power, $P_{l_{pq}}$, and reactive power, $Q_{l_{pq}}$, these can be determined with relations (33), substituting in this case $\hat{I}_l$ with $\hat{I}_{l_{pq}}$.

3.3. Skin Effect Produced by High Harmonics

Having in view that in the current's harmonics frequency spectrum [*qf*, ∞) the skin effect is net, it is possible to utilize relations (7) adequately modified namely

$$\hat{E}_{\mathrm{int}_q}(x) = \frac{\hat{\gamma}_q \hat{I}_{l_q}}{2\sigma}\mathrm{e}^{-\hat{\gamma}\left(\frac{d}{2}-x\right)}, \quad \hat{H}_{\mathrm{int}_q}(x) = \frac{\hat{I}_{l_q}}{2}\mathrm{e}^{-\hat{\gamma}_q\left(\frac{d}{2}-x\right)}, \tag{48}$$

where

$$\hat{I}_{l_q} = \sum_{k=q}^{\infty}\left(1_k I'_{l_k} + \mathrm{j}_k I''_{l_k}\right) \tag{49}$$

and (s. (18))

$$\hat{\gamma}_q = \sum_{k=q}^{\infty}(1_k + \mathrm{j}_k)\alpha_k = \sqrt{\sum_{k=q}^{\infty}\mathrm{j}_k k\omega\mu\sigma}. \tag{50}$$

As regards the Poynting vector hypercomplex modulus, in a point situated on the conducting plate's surface ($x = d/2$), this is given by

$$\hat{S}_{l_q}\left(\frac{d}{2}\right) = \frac{\hat{I}_{l_q}^2}{4\sigma}\hat{\gamma}_q = \frac{\hat{I}_{l_q}^2}{4\sigma}\sum_{k=q}^{\infty}(1_k + \mathrm{j}_k)\alpha_k. \tag{51}$$

Using the same proceeding as in section **3.2** it results

$$P_{l_q} = \frac{\hat{I}_{l_q}^2}{4\sigma}\sum_{k=q}^{\infty}\alpha_k, \quad Q_{l_q} = \frac{\hat{I}_{l_q}^2}{4\sigma}\sum_{k=q}^{\infty}\alpha_k. \tag{52}$$

If expression (19) of attenuation constant, α_k, is taken into account relations (52) become

$$P_{l_q} = \frac{\hat{I}_{l_q}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=q}^{\infty}\sqrt{k}, \quad Q_{l_q} = \frac{\hat{I}_{l_q}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=q}^{\infty}\sqrt{k}. \tag{53}$$

Considering only the $n > q$ first harmonics of the current i_l, *finite* values of these powers are obtained.

3.4. The Global Skin Effect

If the conducting material utilized at realizing the plate is linear it is possible to apply the superposition principle *i.e.*

$$\begin{aligned}\hat{E}_{\text{int}}(x) &= \hat{E}_{\text{int}\,p}(x) + \hat{E}_{\text{int}\,pq}(x) + \hat{E}_{\text{int}\,q}(x),\\ \hat{H}_{\text{int}}(x) &= \hat{H}_{\text{int}\,p}(x) + \hat{H}_{\text{int}\,pq}(x) + \hat{H}_{\text{int}\,q}(x).\end{aligned} \tag{54}$$

Having in view relations (37), (20) and (48) expressions (53) become

$$\begin{aligned}\hat{E}_{\text{int}}(x) = \frac{72\hat{I}_{l_p}}{d\sigma}\cdot\frac{\sum_{k=0}^{p}\left[1_k\left(48 + k^2\omega^2\sigma^2\mu^2d^2x^2\right) + \mathrm{j}_k 2k\omega\sigma\mu\left(12x^2 - d^2\right)\right]}{3456 + p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^4} + \\ + \frac{\hat{\gamma}_{pq}\hat{I}_{l_{pq}}}{2\sigma}\cdot\frac{\operatorname{ch}\hat{\gamma}_{pq}x}{\operatorname{sh}\frac{\hat{\gamma}_{pq}d}{2}} + \frac{\hat{\gamma}_q\hat{I}_{l_q}}{2\sigma}\mathrm{e}^{-\hat{\gamma}_q\left(\frac{d}{2}-x\right)},\end{aligned} \tag{55}$$

respectively

$$\begin{aligned}\hat{H}_{\text{int}}(x) = \frac{24\hat{I}_{l_p}}{d}\cdot\frac{\sum_{k=0}^{p}\left[1_k\left(144 + k^2\omega^2\sigma^2\mu^2d^2x^2\right) + \mathrm{j}_k 6k\omega\sigma\mu\left(d^2 - 4x^2\right)\right]}{3456 + p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^4} + \\ + \frac{\hat{I}_{l_{pq}}}{2}\cdot\frac{\operatorname{ch}\hat{\gamma}_{pq}x}{\operatorname{sh}\frac{\hat{\gamma}_{pq}d}{2}} + \frac{\hat{I}_{l_q}}{2}\mathrm{e}^{-\mathrm{j}\left(\frac{d}{2}-x\right)}.\end{aligned} \tag{56}$$

In the same time

$$\hat{S}_l\left(\frac{d}{2}\right) = \hat{S}_{l_p}\left(\frac{d}{2}\right) + \hat{S}_{l_{pq}}\left(\frac{d}{2}\right) + \hat{S}_{l_q}\left(\frac{d}{2}\right), \tag{57}$$

so, having in view expressions (40), (25), (31) and (51) it results

$$\hat{S}_l\left(\frac{d}{2}\right)=\frac{9I_{l_p}^2}{\sigma d}\cdot\frac{\sum_{k=0}^{p}\left[1_k\left(192+k^2\omega^2\sigma^2\mu^2d^4\right)+\mathrm{j}_k16k\omega\sigma\mu d^2\right]}{3456+p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^4}+\frac{I_{l_{pq}}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\times$$

$$\times\sum_{k=p}^{q}\left(1_k+\mathrm{j}_k\right)k\frac{\sum_{k=0}^{q}1_k\mathrm{sh}\left(\alpha_kd\right)\prod_{\substack{l=0\\l\neq k}}^{q}\mathrm{ch}\left(\alpha_ld\right)-\sum_{k=0}^{q}\mathrm{j}_k\sin\left(\alpha_kd\right)\prod_{\substack{l=0\\l\neq k}}^{q}\cos\left(\alpha_ld\right)}{\prod_{l=p}^{q}\mathrm{ch}\left(\alpha_ld\right)-\prod_{l=p}^{q}\cos\left(\alpha_ld\right)}+ \quad (58)$$

$$+\frac{I_{l_{pq}}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=q}^{\infty}\sqrt{k},$$

where relations (43) were taken into account too.

Relation (57) permits to determine the expressions of active and reactive powers namely

$$P_l\left(\frac{d}{2}\right)=P_{l_p}\left(\frac{d}{2}\right)+P_{l_{pq}}\left(\frac{d}{2}\right)+P_{l_q}\left(\frac{d}{2}\right), \quad (59)$$

respectively

$$Q_l\left(\frac{d}{2}\right)=Q_{l_p}\left(\frac{d}{2}\right)+Q_{l_{pq}}\left(\frac{d}{2}\right)+Q_{l_q}\left(\frac{d}{2}\right). \quad (60)$$

Having in view relations (41), (33_1) and (52_1) expression (59) becomes

$$P_l\left(\frac{d}{2}\right)=\frac{3I_{l_p}^2}{2\sigma d}\cdot\frac{1152+p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^4}{3456+p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^4}+\frac{I_{l_{pq}}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\times$$

$$\times\frac{\sum_{k=p}^{q}\sqrt{k}\mathrm{sh}\left(\alpha_kd\right)\prod_{\substack{l=0\\l\neq k}}^{q}\mathrm{ch}\left(\alpha_ld\right)+\sum_{k=p}^{q}\sqrt{k}\sin\left(\alpha_kd\right)\prod_{\substack{l=0\\l\neq k}}^{q}\cos\left(\alpha_ld\right)}{\prod_{l=p}^{q}\mathrm{ch}\left(\alpha_ld\right)-\prod_{l=p}^{q}\cos\left(\alpha_ld\right)}+\frac{I_{l_q}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=q}^{n}\sqrt{k}. \quad (61)$$

As well taking into account relations (42), (33_2) and (52_2) expression (60) leads to

$$Q_l\left(\frac{d}{2}\right)=\frac{36I_{l_p}^2}{\sigma d}\cdot\frac{p(p+1)\omega\sigma\mu d}{3456+p(p+1)(2p+1)\omega^2\sigma^2\mu^2 d^4}+\frac{I_{l_{pq}}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\times$$

$$\times\frac{\sum_{k=p}^{q}\sqrt{k}\,\mathrm{sh}(\alpha_k d)\prod_{\substack{l=0\\ l\neq k}}^{q}\mathrm{ch}(\alpha_l d)-\sum_{k=p}^{q}\sqrt{k}\sin(\alpha_k d)\prod_{\substack{l=0\\ l\neq k}}^{q}\cos(\alpha_l d)}{\prod_{l=p}^{q}\mathrm{ch}(\alpha_l d)-\prod_{l=p}^{q}\cos(\alpha_l d)}+\frac{I_{l_q}^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=q}^{n}\sqrt{k}. \tag{62}$$

In relations (61) and (62) were considered only the n first harmonics of current i_l.

Disposing of expression (61) of dissipated active power in the plate's length's unit and width's unit and taking into account that the dissipated power in the same plate in d.c., also in her length's unit and width's unit, is

$$P_{0l}=\frac{1}{\sigma d}I_l^2, \tag{63}$$

it is possible to determine the resistance's increase coefficient in a.c.

$$k_a=\frac{P_l}{P_{0l}}. \tag{64}$$

3.4. Particular Case

If the current's fundamental frequency, $f = \omega/2\pi$, is enough high, it is possible to consider that the skin effect is, in his totality, a net one and relations (20) become

$$\hat{E}_{\mathrm{int}}(x)=\frac{\gamma \underline{I}_l}{2\sigma}\mathrm{e}^{-\hat{\gamma}\left(\frac{d}{2}-x\right)},\quad \hat{H}_{\mathrm{int}}(x)=\frac{\hat{I}_l}{2}\mathrm{e}^{-\hat{\gamma}\left(\frac{d}{2}-x\right)}, \tag{65}$$

so that

$$\hat{S}_l\left(\frac{d}{2}\right)=\frac{\hat{\gamma}I_l^2}{4\sigma}\sum_{k=0}^{n}\left(1_k P_k+\mathrm{j}_k Q_k\right), \tag{66}$$

with

$$P_l=\sum_{k=0}^{n}P_k=\frac{I_l^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=0}^{n}\sqrt{k},\quad Q_l=\sum_{k=0}^{n}Q_k=\frac{I_l^2}{4}\sqrt{\frac{\omega\mu}{2\sigma}}\sum_{k=0}^{n}\sqrt{k}, \tag{67}$$

if only the first n harmonics of the current i_l are considered. In this last case is possible to determine the expression of resistance's increase coefficient in a.c., (64), taking into account relations (63) and (67). It results

$$k_a = \frac{1}{4\sqrt{2}}\sqrt{\omega\mu\sigma}\sum_{k=q}^{\infty}\sqrt{k}. \tag{68}$$

In the same time it is possible to determine the hypercomplex wave impedance (Rosman, 2010) too, as ratio of hypercomplex moduli of vectors $\hat{\mathbf{E}}_{\text{int}}(x)$ and $\hat{\mathbf{H}}_{\text{int}}(x)$. Having in view expressions (64) of these ones it results

$$\hat{\zeta}_0 = \frac{\hat{E}_{\text{int}}(x)}{\hat{H}_{\text{int}}(x)} = \sum_{k=0}^{\infty}\left(1_k + \mathrm{j}_k\right)\sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{k\omega\mu}{\sigma}}\mathrm{e}^{\sum_{k=0}^{\infty}\mathrm{j}_k\pi/4}, \tag{69}$$

where relations (18) and (19) were taken into account.

4. Conclusions

The skin effect in a conducting plate having a rectangular section, through which flows a periodical non-harmonic current, is studied. The utilized method is based on a symbolic proceeding which permits to represent a periodical non-harmonic signal through a hypercomplex "image".

While the current's low rank harmonics produce a weak skin effect, the high rank harmonics generate a net skin effect; as regards the medium rank harmonics these ones produce a medium skin effect.

In each of these cases are determined the hypercomplex vectors $\hat{\mathbf{E}}_{\text{int}}(x)$ and $\hat{\mathbf{H}}_{\text{int}}(x)$ moduli, as well the Poynting vector, $\hat{\mathbf{S}}(d/2)$, hypercomplex modulus, in a point situated on the conducting plate's surface. Consequently the active and reactive powers on the plate length's unit and width's unit are determined.

Finally the particular case is studied when the current's fundamental frequency is enough high so that the global skin effect is a net one.

REFERENCES

Mocanu C.I., *Teoria câmpului electromagnetic*. Edit. Did. şi Pedag., Bucureşti, 1984.

Rosman H., *About a Symbolic Representation Method of Periodical Non-Harmonic Signals*. Proc. of the 6th Internat. Conf. on Electr. a. Power Engnmg., EPE 2010, **I**, 227-229 .

Rosman H., *Penetration of the Periodic Non-Sinusoidal Electromagnetic Field into Conducting Half-Space*. Rev. Roum. Sci. Techn., s. Électrot. et Énerg., **24**, *3*, 389-395 (1979).

Rosman H., *Reprezentarea simbolică a puterilor în regim permanent periodic nesinusoidal.* Bul. Inst. Politehnic, Iaşi, **VI (X)**, *3-4*, 261-270 (1960).

Rosman H., *Waves Hypercomplex Impedance in Case of Penetration of an Electromagnetic Periodic Non-Harmonic Field into Conducting Semi-Infinite Space*. Bul. Inst. Politehnic, Iaşi, **LVI (LX)**, *2*, s. Electrot., Energ., Electron., 9-13 (2010).

Rozenfeld B.A., *Symbolic Method and Vectorial Diagrams for Non-Sinusoidal Currents* (in Russian). Tr. sem. po vekt. i tenz. anal., **7**, 381-387 (1949).

Ryžik J.M., Gradshtein J.S., *Integrals, Sums and Derivatives* (in Russian). Third Edition, GITTL, Moscow-Leningrad, 1951.

EFECTUL PELICULAR ÎNTR-O PLACĂ CONDUCTOARE DREAPTĂ, AVÂND SECȚIUNEA DREPTUNGHIULARĂ, ÎN REGIM PERMANENT PERIODIC NEARMONIC

(Rezumat)

Folosind o metodă simbolică bazată pe reprezentarea semnalelor periodice nearmonice prin „imagini” hipercomplexe, se determină modulii vectorilor hipercomplecşi de stare $\hat{\mathbf{E}}_{\mathrm{int}}$ şi $\hat{\mathbf{H}}_{\mathrm{int}}$ într-un punct situat în interiorul unei plăci conductoare drepte, de secțiune dreptunghiulară, prin care circulă un curent periodic nearmonic. De asemenea se determină modulul vectorului hipercomplex al lui Poynting într-un punct situat pe suprafața plăcii conductoare, permițând calculul puterilor activă şi reactivă în unitatea de lungime a plăcii conductoare şi în unitatea de grosime a acesteia.