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ON THE FRACTAL DIMENSION OF THE SIGNALS GENERATED BY SOME NONLINEAR CIRCUITS

BY

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Abstract. A relatively new method of investigation, the visibility graph, allows us to compute the fractal dimension of a chaotic signal using the time series that it represents. The method was applied to some simple nonlinear circuits, known for their chaos-generating properties that were not previously characterized by means of the fractal dimension.

Key words: visibility graph; “jerk” equation; time series; chaos.

1. Introduction

The nonlinear circuits that exhibit chaotic behaviour were studied from several points of view: from their minimalist implementation using electronic components (Siriburanon *et al.*, 2010; Srisuchinwong *et al.*, 2012) to their properties as generators of chaotic signals characterized by Lyapunov exponents (Radwan *et al.*, 2003), phase portraits and rarely, different measures of their dimension, usually the Kaplan-Yorke one as a consequence to the knowledge of the Lyapunov exponents.

An interesting and quite large class of circuits of this kind was described and studied in (Sprott, 2000) and later on in (Sprott, 2011). They were

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based on the so called "jerk" equation, sometimes named model, as in (Srisuchinwong *et al*., 2012). The name comes from mechanics and denotes the third derivative of the space (or the first derivative of the acceleration) seen as a time function. It was introduced and defined by Schot in (Schot, 1978).

There are not so many criteria that allow choosing among different chaos generators: *e.g*. previous work in (Sprott, 2000) is intended mostly as a review of possibilities for a generic jerk differential equation and therefore the main topic lays in describing the circuits that are associated, their conform behaviour to the theoretical background and eventually, computing Lyapunov exponents for certain cases of interest. That is why it seems interesting enough to try the characterization of the generated signals by fractal dimension, a typical measure for chaotic signals that is seldom used, mostly because of the simple way of computing another different dimension, the Kaplan-Yorke one, when the Lyapunov coefficients are known (Frederickson *et al*., 1983). This becomes more encouraging since the introduction of the concept of visibility graph, a tool that permits mapping of the time series to associated graphs and consequently, to use the graph theory to characterise the time series from different points of view (Lacasa, 2008).

Knowing the fractal dimension of the signal that is generated by a nonlinear circuit is important when it is intended for random number generators, like the one described in (Yalcin *et al*., 2004) and when one is interested to evaluate the self-similarity of the time series that represents the signal, trying to determine how much chaos and how much noise is embedded. It also may be a good answer to the question "what is the difference between chaotic circuits that are implemented according to the same type of equation?" since those that were already devised for such purposes all have positive Lyapunov exponents and from this point of view, exhibit chaotic behaviour in a way or another. In this manner, a supplementary criterion may be used. Also characterizing the self-similarity of a signal allows us to use two different scales in order to get rid of the noise in one of them if the existence of the self-similarity was previously proven by means of the fractal dimension.

The paper is organised as follows: first there is a short presentation of the visibility graph and its main definitions, intended for the newcomers in the field, next the algorithm that allows computing the fractal dimension by means of the visibility graph is reviewed for short. The third section is devoted to the general description of the circuits that are implemented according to the jerk equation and finally the results concerning the fractal dimension of the chosen circuits are presented, along with several conclusions in the end.

2. The Visibility Graph – a Link between Time Series and Graphs

The visibility graph was defined in (Lacasa, 2008a, b) and later on developed as horizontal visibility graph (Luque *et al*., 2009). It was initially intended to explore the possibility of using the already well defined tools of the graph theory in order to extract features that were hard to compute by other means, especially when dealing with nonlinear phenomena. When using the tools for chaos characterization there were several instances that benefited from the method: EEG signal characterization and feature extraction (Zhu *et al.*, 2012), autism (Ahmadlou *et al.*, 2012), economics (Wang *et al.*, 2012), social (Fan *et al.*, 2012) and earth sciences (Telesca *et al.*, 2012) to quote just a few of the papers that appeared during the same year.

The rules that map a time series into its associated visibility graph are the following: 1) each element of the time series uniquely represents a node of the visibility graph and 2) two nodes are connected by an edge if and only if they are able to ”see” each other. The definitions are illustrated in Fig. 1 for a time series denoted $\{x_n\}$, and a few generically chosen points, from i to i + 6. It is worth noticing in the graph representation that the horizontally drawn lines correspond to the obvious fact that consecutive samples see each other and as of matter of consequence, successive nodes are linked together by default.

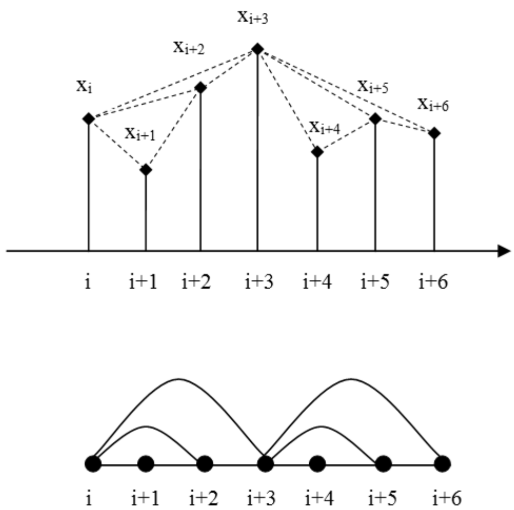


Fig. 1 – The rules that generate the visibility graph of the $\{x_n\}$ time series and its associated graph.

The algorithm involved in mapping the time series to its associated visibility graph may be formulated analytically for the second rule, according to Fig. 2, since the number of nodes equals the number of the samples of the time series and this defines them from the beginning.

The equation of the line that passes through two different points of coordinates (i, x_i) and (j, x_j) with generic variables (p, x) is the following (see Fig. 2):

$$x - x_i = \frac{x_j - x_i}{j - i}(p - i)\,. \tag{1}$$

Obviously, if the nodes i and j see each other, for any value of k between i and j, the following inequality holds true for the coordinates (k, x_k):

$$x_k < x_i + \frac{x_j - x_i}{j - i}(k - i). \tag{2}$$

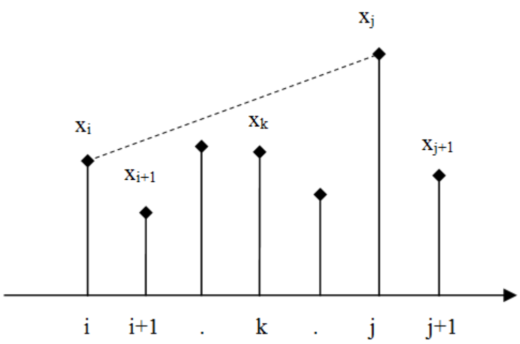


Fig. 2 – The dotted line, equation (1), that links two samples of coordinates (i, x_i) and (j, x_j) that see each other.

If two nodes see each other (*i.e.* inequality (2) is true for any node in between), then the two nodes are connected by an edge. For every graph there is an adjacency matrix (in this case symmetrical) for which we can define accordingly, for each $i \neq j$:

$$a_{i,j} = a_{j,i} = \begin{cases} 1 & (i,j) \text{ connected;} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The main diagonal of the adjacency matrix is filled with zeroes due to the above rule. Also, the elements that are adjacent to the main diagonal, both in the lower part and in the upper part of the matrix are ones since two consecutive nodes are visible to each other:

$$a_{i,i-1} = a_{i,i+1} = 1. \tag{4}$$

Essentially, the above mapping allows determining the fractal dimension directly from the adjacency matrix if one defines in advance the order of the nodes. This is the number of edges that are connected to a certain node. This definition allows computing the probability of the distribution of the edges for the nodes of the graph which may be expressed by (Lacassa *et al.*, 2008):

$$P(k) = k^{-\lambda}, \tag{5}$$

with k denoting the order of the node and λ the power of the scale-freeness.

According to (Lacassa *et al.*, 2008) the fractality of the studied time series is expressed by the slope of the line defined by the graph of P(k) versus k^{-1}, both represented in logarithmic scales, which represents the fractal dimension of the time series, FD. This way of computing the fractal dimension is far less difficult than the usual algorithms (Higuchi, Katz or Petrosian, see (Esteller *et al.*, 2001) for a thorough survey).

In this study, the algorithm of Lacassa was slightly adapted from the following points of view: 1) since every node of the visibility graph has an edge linked with its neighbor, the case $k \leq 2$ was not taken into account when the slope of the line in logarithmic coordinates (i.e. the fractal dimension, FD) was computed and 2) as stated in (Ahmadlou *et al.*, 2012), considering only samples of the time series taken at different time intervals, computing for each instance the fractal dimension (for the new, smaller time series) and considering the average, the outcome is a better estimate of the fractal dimension FD, than each individual result. This seems reasonable, even from statistical point of view.

There is another decision that has to be made, according to the algorithm above: where to stop with the time interval so that there is an obvious condition that allows exiting the algorithm. The natural one, to stop at the end of the time series is quite impractical when dealing with long time series, so the one suggested in (Ahmadlou *et al.*, 2012) may be taken into account: the algorithm stops when two consecutive values for the fractal dimension are different by less than a certain amount, denoted ε. During the simulations a value of $\varepsilon = 10^{-2}$ was considered satisfactory, providing a good compromise between the precision and the computing time needed to complete the algorithm on a usual personal computer.

3. The Jerk Equation and the Nonlinear Circuits that Implement It

The name of the equation when used in electrical engineering was introduced in (Sprott, 1997) and is essentially a differential equation of the third degree:

$$F(\dddot{x}, \ddot{x}, \dot{x}, x) = 0 \tag{6}$$

In what follows we shall use the form and the notations from (Sprott, 2000), where the equation is written:

$$\dddot{x} + A \cdot \ddot{x} + \dot{x} = G(x) \tag{7}$$

with $A = 0.6$, while $G(x)$ is a nonlinear function of x. To solve it in an easier way, the above equation may be written as a system of first degree differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y; \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = z; \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = -Az - y + G(x). \end{cases} \tag{8}$$

There are several possibilities to generate chaos choosing a certain function for $G(x)$. Most of them were presented and implemented in (Sprott, 2000) and further developed and studied in (Siriburanon *et al.*, 2010). To illustrate the method and to allow the possibility of comparisons with other results, the following six different $G(x)$ functions were chosen to be characterised by the fractal dimension FD computed by means of the visibility graph:

$$G_1(x) = 1 - |x|, \tag{9}$$

$$G_2(x) = -2.7\sin x, \tag{10}$$

$$G_3(x) = -2.7\cos x, \tag{11}$$

$$G_4(x) = 2.2(x - 2\tanh x), \tag{12}$$

$$G_5(x) = 1 - 6\max(x,0), \tag{13}$$

$$G_6(x) = 1.2x - \mathrm{sign}(x). \tag{14}$$

One important issue when one tries to compute the fractal dimension (FD) by means of the above presented method is the fact that due to the transient process at the beginning of the simulation, materialized in a somewhat slow evolution, there is a certain possibility to encounter a very long process until the precision imposed is reached. To overcome this, after all the values were generated, the time series was cut in half and only the second part of it was used in computations. The time considered for investigation was one second.

Another reason to use the visibility graph when computing the fractal dimension is that there is no need for a huge number of samples (some authors use just a few hundred). In this case just 2000 samples were used to be sure that the desired precision is reached for the fractal dimension; that means the whole time series had 4000 samples. Other values, both smaller and greater, were used but they did not prove to determine decisive changes in the final results.

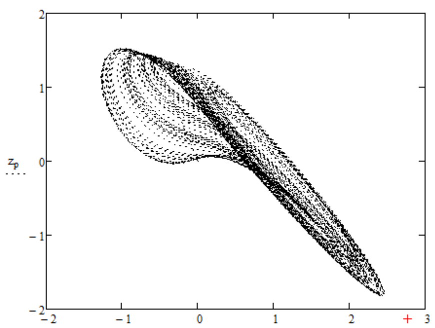
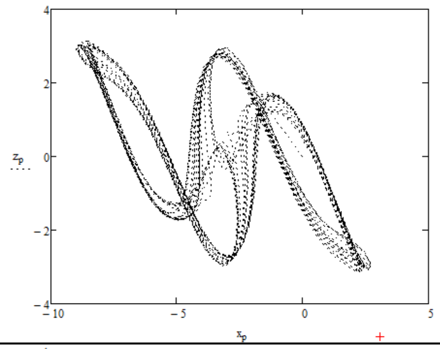
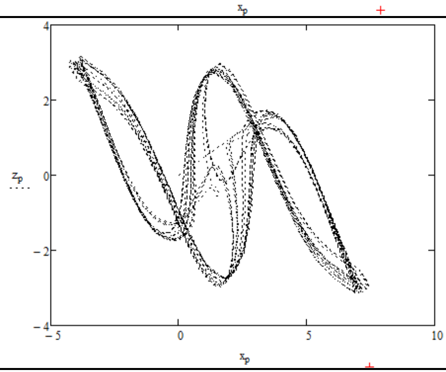
4. Results and Conclusions

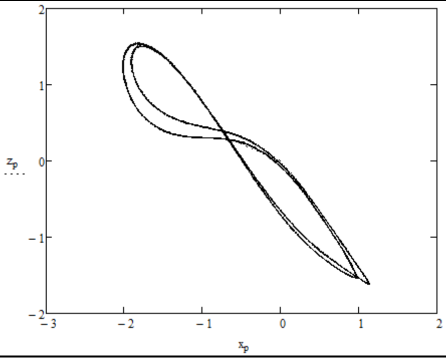
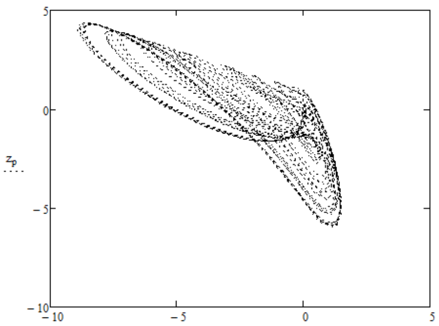
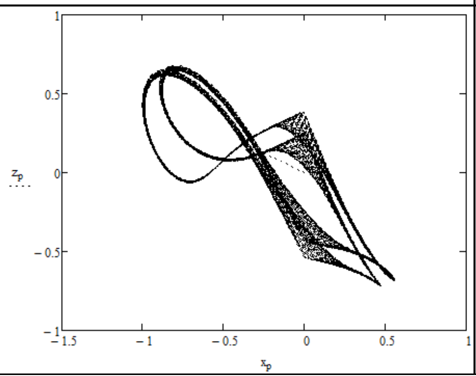
As already acknowledged, the fractal dimension is supposed to be smaller than the Kaplan-Yorke dimension computed when the largest Lyapunov exponents are known. Fortunately the largest Lyapunov exponents for the chosen functions are known (Sprott, 2000) and therefore it is possible to compare the outcome of the visibility graph method with an already computed value, the Kaplan-Yorke dimension.

The results are presented in the Table 1.

Table 1

The Kaplan-Yorke Dimension vs. the Fractal Dimension

$z = f(x)$	$G(x)$	K-Y	FD
	$G_1(x) = 1 - \lvert x \rvert$	2.057	2.19
	$G_2(x) = -2.7\sin x$	2.103	1.891
	$G_3(x) = -2.7\cos x$	2.103	1.594

	$G_4(x) = 2.2(x - 2\tanh x)$	2.269	0.974
	$G_5(x) = 1 - 6\max(x,0)$	2.134	2.29
	$G_6(x) = 1.2 - \mathrm{sign}(x)$	2.523	1.184

There are two cases in which the computed fractal dimension is slightly greater than its Kaplan-Yorke counterpart, for the first and the fifth functions and this is unusual. The main guess to explain this is that the transient process was not entirely finished; this assumption was made knowing the numbers of iterations (*i.e.* fractal dimensions) that were carried out to reach the 0.01 difference imposed between two consecutive ones. Usually this number was five to seven but in the above cases these figures were almost double. For the sake of equivalence, the time and the number of samples were kept the same for all cases and hence the differences.

It is also easy to observe from the graphs that there is a closed link between the computed fractal dimension and the evolution of the $z = f(x)$ function: a greater “fill” of the surface of the graph is reflected into a grater fractal dimension. This observation gives a hint concerning the candidates for random numbers generators among the studied circuits.

To conclude, it is worth noticing a few facts: 1) the number of the samples of the time series that represents the signal may be lower than in most other computations dealing with the same subject; 2) one must be sure that the transients were discarded from the chosen values of the time series to obtain a valid result; 3) the method of the visibility graph is able to be a good indicator for the chaotic behaviour of a certain nonlinear circuit through the fractal dimension; 4) it is also possible to discern between noise and chaos utilizing the computing values but this is beyond the purpose of this paper.

Further work may reveal the links between the fractal dimension and the influence of the components of a circuit that implements the jerk equation in different cases of implementation and the way to extract noise from a chaotic signal.

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ASUPRA DIMENSIUNII FRACTALE A SEMNALELOR GENERATE DE UNELE CIRCUITE NELINIARE

(Rezumat)

Sunt prezentate noi rezultate obţinute în ceea ce priveşte caracterizarea unor semnale haotice cu ajutorul dimensiunii fractale. Aceasta este calculată prin metoda grafului de vizibilitate care se dovedeşte expeditivă şi exactă în condiţiile în care este luat în considerare regimul tranzitoriu iniţial, inerent funcţionării oricărui circuit.