

SOME ASPECTS REGARDING THE QUALITY FACTOR OF CYLINDRICAL WAVEGUIDE RESONATORS

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Abstract. The paper presents a study of the quality factor, Q , for cylindrical resonators with metal walls. The dielectric losses and the Joule losses in the walls are taken into account in establishing the mathematical expressions for Q . A numerical simulation is conducted in order to investigate the influence of real lossy dielectrics, such as ice, dust and distilled water on the value of Q and the resonance frequency.

Keywords: cavity resonators; quality factor; numerical integration.

1. Introduction

Waveguide resonators are passive microwave devices used for amplifying high frequency signals, for measurements of frequency and dielectric constants, or as microwave filters. Two geometries are currently used: the rectangular and the cylindrical resonator.

The important factors that characterize the waveguide resonator are the quality factor, Q , and the power to field conversion efficiency (Annino, 2006). Since both these quantities, as well as the resonant frequency, depend on the resonator geometry and dimensions, an exact determination, based on analytical

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solutions of the electromagnetic field problem, when possible, is of paramount importance, especially in the analysis of the Q factor stability to geometrical or material imperfections.

The general theory of waveguide resonators is well documented and analytical expressions for the electromagnetic field components are established in TM and TE modes for resonant cavities with uniform, ideal dielectrics (Miner, 1996).

Some papers (Mohammad-Taheri, 1990) establish and discuss the quality factor of cylindrical resonators with metal walls, loaded with a hollow or full dielectric. In this case the magnetic field components are expressed as a series of finite element basis functions (Lagrange polynomials) which enable the determination of an analytical solution.

In (Savin *et al.*, 2014) the quality factor of a tunable cylindrical resonator (with a mobile base), partially filled with a cylindrical dielectric, is studied experimentally using the mode matching technique and a microwave vector analyzer. Simulation results, based on the finite integration technique (FIT), which uses a discretization of the domain in order to solve Maxwell's equations in integral form, are shown.

This paper presents a theoretical study of the cylindrical resonator filled with an ideal or lossy dielectric. The established mathematical expressions enable the calculation of the quality factor and the investigation of the influence of dielectric parameters on its performance.

2. Problem Formulation. Electric and Magnetic Field Solutions

The schematic model of a cylindrical resonator with metal walls and homogeneous dielectric is presented in Fig. 1.

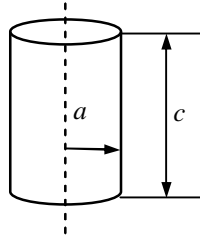


Fig. 1 – Cylindrical resonator.

Excluding from the start the hybrid modes (which are in fact a superposition of simple modes), the cavity can be excited in TM or TE mode.

Considering first the case of the *TM mode*, the electric and magnetic field components in the cylindrical coordinate system have the expressions (Miner, 1996):

$$\underline{E}_z(r, \theta, z) = E_0 J_n \left(\frac{p_{nl}}{a} r \right) \cos(n\theta) \cos(\beta z), \quad (1)$$

$$\underline{E}_r(r, \theta, z) = -\frac{\beta a}{p_{nl}} E_0 J_n' \left(\frac{p_{nl} r}{a} \right) \cos(n\theta) \sin(\beta z), \quad (2)$$

$$\underline{E}_\theta(r, \theta, z) = -\frac{n \beta a^2}{p_{nl}^2 r} E_0 J_n \left(\frac{p_{nl} r}{a} \right) \sin(n\theta) \sin(\beta z), \quad (3)$$

$$\underline{H}_r(r, \theta, z) = -\frac{j \omega \varepsilon n a^2}{p_{nl}^2 r} E_0 J_n \left(\frac{p_{nl} r}{a} \right) \sin(n\theta) \cos(\beta z), \quad (4)$$

$$\underline{H}_\theta(r, \theta, z) = -\frac{j \omega \varepsilon a}{p_{nl}} E_0 J_n' \left(\frac{p_{nl} r}{a} \right) \cos(n\theta) \cos(\beta z). \quad (5)$$

The resonance frequency, ω_r , has the expression:

$$\omega_r = \frac{1}{\sqrt{\mu \varepsilon}} \sqrt{\left(\frac{q \pi}{c} \right)^2 + \left(\frac{p_{nl}}{a} \right)^2}. \quad (6)$$

In relations (1)...(5) $J_n(kr)$ is the n -th order Bessel function and p_{nl} is its l -th root, $J_n(p_{nl}) = 0$. The phase constant β is:

$$\beta = \frac{q \pi}{c}, \quad q = 0, 1, 2, \dots \quad (7)$$

so that the mode is called TM_{nlq} . The lowest frequency modes that can be obtained are TM_{010} , TM_{011} and TM_{110} .

In the case of the *TE mode* the electric and magnetic field components have the expressions (Miner, 1996):

$$\underline{H}_z(r, \theta, z) = H_0 J_n \left(\frac{p_{nl} r}{a} \right) \cos(n\theta) \sin(\beta z), \quad (8)$$

$$\underline{H}_r(r, \theta, z) = \frac{\beta}{p_{nl}/a} H_0 J_n' \left(\frac{p_{nl} r}{a} \right) \cos(n\theta) \cos(\beta z), \quad (9)$$

$$\underline{H}_\theta(r, \theta, z) = -\frac{n \beta}{(p_{nl}/a)^2 r} H_0 J_n \left(\frac{p_{nl} r}{a} \right) \sin(n\theta) \cos(\beta z), \quad (10)$$

$$\underline{E}_r(r, \theta, z) = -\frac{j \omega \mu n}{(p_{nl}/a)^2 r} H_0 J_n \left(\frac{p_{nl} r}{a} \right) \sin(n\theta) \sin(\beta z), \quad (11)$$

$$\underline{E}_\theta(r, \theta, z) = -\frac{j \omega \mu}{p_{nl}/a} H_0 J_n' \left(\frac{p_{nl} r}{a} \right) \cos(n\theta) \sin(\beta z). \quad (12)$$

In relations (8),...(12) p'_{nl} is the l -th order zero of the derivative of $J_n(kr)$, i.e. $J'_n(p'_{nl}) = 0$. The resonance frequency has the same expression as in the TM mode, with the difference that p_{nl} is substituted by p'_{nl} . The modes are called TE_{nlq} and the lowest frequency that can be excited is TE_{011} .

3. Quality Factor of the Cylindrical Resonator

The quality factor of a waveguide resonator is defined as

$$Q = \frac{\omega_r W_{\max}}{P}, \quad (13)$$

where: W_{\max} is the maximum electromagnetic energy stored in the cavity. Since the maximum electric energy is equal to the maximum magnetic energy at resonance, W_{\max} can be calculated with one of the following relations:

$$W_{\max} = \frac{1}{2} \iiint_{v_{\Sigma}} \varepsilon E_{\max}^2 dv, \quad W_{\max} = \frac{1}{2} \iiint_{v_{\Sigma}} \mu H_{\max}^2 dv. \quad (14)$$

For of an ideal dielectric P represents the power losses due to the currents induced in the metal walls (Joule effect), P_J . If the dielectric in the cavity is real, the losses caused by cyclic electric polarization, P_{diel} , must be also taken into account, so that in this case

$$P = P_J + P_{\text{diel}}. \quad (15)$$

For the high frequencies representing the working range of the waveguide (GHz), the Joule losses inside the metal walls can be calculated with the following formula (Miner, 1996):

$$P_J = \iint_{\Sigma} \Re \left\{ \underline{E}_t \times \underline{H}_t^* \right\} \cdot \underline{n} dA = \frac{1}{\delta \sigma} \iint_{\Sigma} |\underline{H}_t|^2 dA, \quad (16)$$

where: δ is the penetration depth, σ is the wall conductivity and \underline{E}_t , \underline{H}_t are the electric and magnetic field components, tangential to the wall surface Σ .

Let us consider again the two basic modes, TM and TE. Using the relations presented in the previous section, the mathematical expressions for Q can be established.

For the TM_{nlq} mode the maximum magnetic energy is calculated with the relation:

$$W_{\max} = W_{mg \max} = 2 \iiint_{v_{\Sigma}} \frac{\mu H^2}{2} r dr d\theta dz, \quad (17)$$

where: $H^2 = \left\{ \underline{H} \cdot \underline{H}^* \right\}$ and \cdot represents the dot product. The result depends on whether $n \neq 0$ or $n = 0$, and also $q \neq 0$ or $q = 0$. In the general case, for $n \neq 0$ and $q \neq 0$, the calculation can be done only using a numerical integration technique

(Angot, 1961). Using the expressions for \underline{H}_r and \underline{H}_θ and introducing the notations

$$I_1 = \int_0^a \frac{1}{r} J_n^2 \left(\frac{p_{nl}}{a} r \right) dr, \quad n \neq 0, \quad (18)$$

$$I_4 = \int_0^a J_n \left(\frac{p_{nl}}{a} r \right) \cdot J_{n+1} \left(\frac{p_{nl}}{a} r \right) dr, \quad (19)$$

$$I_5 = \int_0^a r J_{n+1}^2 \left(\frac{p_{nl}}{a} r \right) dr, \quad (20)$$

$$I_6 = \int_0^a r J_n^2 \left(\frac{p_{nl}}{a} r \right) dr, \quad (21)$$

$$I_2 = \int_0^a r \left[J_n' \left(\frac{p_{nl}}{a} r \right) \right]^2 dr = \frac{n^2 a^2}{p_{nl}^2} I_1 - \frac{2na}{p_{nl}} I_4 + I_5, \quad (22)$$

the final expression of $W_{mg \max}$ in TM mode can be written as:

$$W_{mg \max} = \begin{cases} \frac{\pi c \mu \omega^2 \varepsilon^2 a^2}{2 p_{nl}^2} E_0^2 \left(2 \frac{n^2 a^2}{p_{nl}^2} I_1 - 2 \frac{na}{p_{nl}} I_4 + I_5 \right), & n \neq 0, \quad q \neq 0, \\ \frac{\pi c \mu \omega^2 \varepsilon^2 a^2}{p_{0l}^2} E_0^2 \frac{a^2}{2} J_1^2(p_{0l}), & n = 0, \quad q \neq 0, \\ \frac{\pi c \mu \omega^2 \varepsilon^2 a^2}{p_{0l}^2} E_0^2 a^2 J_1^2(p_{0l}), & n = 0, \quad q = 0. \end{cases} \quad (23)$$

The Joule losses in the metal walls of conductivity σ , based on (16), have thus the following expressions:

$$P_J = \begin{cases} \frac{1}{\delta \sigma} \cdot \frac{\omega^2 \varepsilon^2 a^2}{p_{nl}^2} \pi E_0^2 \left(\frac{ac}{2} J_{n+1}^2(p_{nl}) + \frac{2n^2 a^2}{p_{nl}^2} I_1 + 2I_2 \right), & n \neq 0, \quad q \neq 0, \\ \frac{1}{\delta \sigma} \cdot \frac{\omega^2 \varepsilon^2 a^2}{p_{0l}^2} \pi E_0^2 (ac + 2a^2) J_1^2(p_{0l}), & n = 0, \quad q \neq 0, \\ \frac{1}{\delta \sigma} \cdot \frac{\omega^2 \varepsilon^2 a^2}{p_{0l}^2} \pi E_0^2 (2ac + 2a^2) J_1^2(p_{0l}), & n = 0, \quad q = 0, \end{cases} \quad (24)$$

where the previously defined notations for the integrals I_1, \dots, I_6 have been used.

If the dielectric inside the cavity is supposed to present losses, with the complex permittivity

$$\underline{\varepsilon} = \varepsilon' - j\varepsilon'' , \quad (25)$$

then the power dissipated in the lossy dielectric may be evaluated using the relation

$$P_{\text{diel}} = \iiint_{v_{\Sigma}} \omega_r \varepsilon'' E^2 dv . \quad (26)$$

In the case of complex permittivity, the complex propagation constant, $\underline{\gamma}$, is

$$\underline{\gamma} = \alpha_d + j\beta . \quad (27)$$

where: α_d is the attenuation due to dielectric losses and β is the phase constant, having the expressions (Petrescu, 2002):

$$\alpha_d = \omega \sqrt{\frac{\mu \varepsilon'}{2}} \sqrt{\frac{k_c^2}{\omega^2 \mu \varepsilon'} - 1 + \sqrt{\left(1 - \frac{k_c^2}{\omega^2 \mu \varepsilon'}\right)^2 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2}} , \quad (28)$$

$$\beta = \omega \sqrt{\frac{\mu \varepsilon'}{2}} \sqrt{1 - \frac{k_c^2}{\omega^2 \mu \varepsilon'} + \sqrt{\left(1 - \frac{k_c^2}{\omega^2 \mu \varepsilon'}\right)^2 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2}} .$$

The constant k_c is:

$$k_c = \begin{cases} \frac{p_{nl}}{a} , & \text{TM mode,} \\ \frac{p_{nl}}{a} , & \text{TE mode.} \end{cases} \quad (29)$$

Using the previously defined integrals, the additional power dissipation introduced by the lossy dielectric in the TM mode has the expression:

$$P_{\text{diel}} = \begin{cases} \frac{\pi c}{2} \omega_r \varepsilon \text{tg} \delta E_0^2 \left(I_6 + \frac{\beta^2 a^2}{p_{nl}^2} I_2 + \frac{n^2 \beta^2 a^4}{p_{nl}^4} I_1 \right) , & n \neq 0, q \neq 0, \\ \pi c \omega_r \varepsilon \text{tg} \delta E_0^2 \left(I_6 + \frac{\beta^2 a^2}{p_{0l}^2} I_2 \right) , & n = 0, q \neq 0, \\ 2\pi c \omega_r \varepsilon \text{tg} \delta E_0^2 I_6 , & n = 0, q = 0. \end{cases} \quad (30)$$

For the TE_{nlq} mode the maximum electric energy is:

$$W_{\text{max}} = W_{el \text{ max}} = 2 \iiint_{v_{\Sigma}} \frac{\mu E^2}{2} dv = \iiint_{v_{\Sigma}} \varepsilon \left| \underline{E} \cdot \underline{E}^* \right| r dr d\theta dz . \quad (31)$$

Using the expressions for $\underline{E}_r(r, \theta, z)$, $\underline{E}_\theta(r, \theta, z)$ given by relations (11), (12) and the notations for I_1, \dots, I_6 , where the constant p_{nl} is replaced by p'_{nl} , the final expression for W_{max} in TE mode is :

$$W_{e \max} = \begin{cases} \frac{\pi c}{2} \cdot \frac{\varepsilon \omega^2 \mu^2}{\left(\frac{p_{nl}}{a}\right)^2} H_0^2 \left[\frac{n^2}{\left(\frac{p_{nl}}{a}\right)^2} I_1 + I_2 \right], & n \neq 0, q \neq 0, \\ \pi c \frac{\varepsilon \omega^2 \mu^2}{\left(\frac{p_{nl}}{a}\right)^2} H_0^2 I_5, & n = 0, q \neq 0. \end{cases} \quad (32)$$

The Joule losses in the metal walls, calculated using (16), have the final expression:

$$P_J = \begin{cases} \frac{H_0^2}{\delta \sigma} \left\{ \frac{\pi a c}{2} J_n^2(p_{nl}) \left(\frac{n^2 \beta^2 a^2}{\left(\frac{p_{nl}}{a}\right)^4} + 1 \right) + \frac{2\beta^2 \pi}{\left(\frac{p_{nl}}{a}\right)^2} \left[\frac{n^2}{\left(\frac{p_{nl}}{a}\right)^2} I_1 + I_2 \right] \right\}, & n \neq 0, \\ \frac{H_0^2}{\delta \sigma} 2\pi a \left[\frac{c}{2} J_0^2(p_{0l}) + \frac{2\beta^2 a}{\left(p_{0l}\right)^2} I_5 \right], & n = 0. \end{cases} \quad (33)$$

Similarly, if the dielectric, supposed to be uniform, is lossy, the power dissipated by cyclic polarization is:

$$P_{diel} = \begin{cases} \frac{\omega_r^3 \varepsilon \operatorname{tg} \delta \mu^2 H_0^2}{\left(\frac{p_{nl}}{a}\right)^2} \cdot \frac{\pi c}{2} \left[\frac{n^2 a^2}{\left(\frac{p_{nl}}{a}\right)^2} I_1 + I_2 \right], & n \neq 0, \\ \frac{\omega_r^3 \varepsilon \operatorname{tg} \delta \mu^2 H_0^2}{\left(\frac{p_{0l}}{a}\right)^2} \pi c I_5, & n = 0. \end{cases} \quad (34)$$

The quality factor is thus given by the expression:

$$Q = \frac{\omega_r W_{\max}}{P_J + P_{diel}}. \quad (35)$$

4. Results and Discussions

The quality factor of a cylindrical resonator with metal walls, containing a lossy or lossless dielectric, has been calculated for several TE and TM modes. The integrals I_1, \dots, I_6 , for which there is no direct analytical, closed form solution, have been calculated by numerical integration using the MATLAB routine *quad*. A precursory study of the accuracy of the available MATLAB routines for single valued scalar function integration (*quad*, *quadgk*, *quadl*) lead to the conclusion that, for the integrals in question, all routines gave similar results. In the end the routine *quad*, more frequently used, was chosen for this study.

The numerical values used in calculations were: $a = 1$ cm, $c = 2$ cm, $\sigma = 5.96 \times 10^7$ S/m (copper). Three cases of lossy dielectric were considered, their dielectric constant and loss angle being given in Table 1 for 3 GHz (<http://www.rfcafe.com/references/electrical/dielectric-constants-strengths.htm>).

Table 2 presents the results obtained for the TM mode when the cavity is empty (air) and for the three lossy dielectrics indicated in Table 1.

Table 1
Electrical Parameters for the Lossy Dielectrics

Dielectric	ϵ_r	$\text{tg}\delta$
a) ice	3.2	0.0009
b) water (distilled)	80.4	0.157
c) soil (sandy)	3.55	0.0062

Table 2
Quality Factor and Resonance Frequency in TM mode

	Dielectric	$n = 0$				$n = 1$			
		$q = 0$		$q = 1$		$q = 0$		$q = 1$	
		$l = 1$	$l = 2$	$l = 1$	$l = 2$	$l = 1$	$l = 2$	$l = 1$	$l = 2$
Q	air	10,958	16,602	8,982	12,696	10,375	14,037	10,788	14,210
	ice	978.3	1,019.8	953	994.7	971.8	1004	976.6	1,006
	water	6.35	6.4	6.23	6.06	5.89	5.9	6.14	6
	sandy soil	158.0	159.2	157	158.5	157.4	158.3	158	158.8
f_r (GHz)	air	11.4	26.3	13.7	27.4	18.3	33.5	19.7	34.3
	ice	6.4	14.7	7.66	15.3	10.22	18.72	11.0	19.2
	water	1.28	2.9	1.52	3.05	2.04	3.73	2.20	3.82
	sandy soil	6.09	13.9	7.27	14.54	9.71	17.77	10.49	18.22

In Table 3 the results for the TE mode and the four types of dielectric filling are presented.

Table 3
Quality factor and resonance frequency in TE mode

	Dielectric	$n = 0$				$n = 1$			
		$q = 1$		$q = 2$		$q = 1$		$q = 2$	
		$l = 1$	$l = 2$	$l = 1$	$l = 2$	$l = 1$	$l = 2$	$l = 1$	$l = 2$
Q	air	21,564	28,414	23,590	29,382	11,625	24,118	14,259	25,446
	ice	1,039	1,056	1,045	1,058	985	1,046	1,006	1,049
	water	6.4	6.4	6.4	6.4	6.35	6.4	6.4	6.4
	sandy soil	159.7	160.1	159.8	160.1	158.2	159.8	158.8	160
f_r (GHz)	air	19.77	34.32	23.65	36.70	11.55	26.53	17.38	29.54
	ice	11.05	19.9	13.22	20.5	6.45	14.83	9.71	16.5
	water	2.20	3.82	2.63	4.09	1.28	2.96	1.93	3.29
	sandy soil	10.49	18.22	12.55	19.48	6.13	14	9.22	15.68

Analyzing the data in Tables 2 and 3 some conclusions, outlined below, can be formulated.

1) Very high quality factors are obtained when the cylindrical resonator is empty. The values of Q are significantly higher for the TE mode, and increase for increasing values of n , l and q . From this perspective and taking into account that the lowest order (fundamental) modes are preferred in practice due to the simplicity of excitation procedure, the most frequently used mode in a cylindrical resonator is TE_{011} (Annino, 2006).

2) When a lossy dielectric fills the resonator the quality factor, as well as the resonance frequency, have a dramatic decrease. The values of Q in the TE and TM modes now become comparable, almost equal in some cases. The decrease is caused by the loss factor, $\text{tg}\delta$, and is most evident for the dielectric with the highest losses (distilled water). The dielectric constant, ϵ_r , also plays an important part in the decrease of Q and f_r .

3) The three examples of lossy dielectrics were chosen so as to simulate an accidental penetration of the resonator with water (liquid or ice) or sandy soil. The data were considered for the highest frequency available in literature, $f = 3$ GHz, which is lower in many cases than that obtained for f_r . However, it is well-known that for lossy dielectrics the loss angle increases with frequency, so that it may be expected that Q has even lower values than those obtained in Tables 2 and 3.

4) These results show that the presence of water or dust in the cavity (even if not completely filled) may be easily detected by the sudden decrease of the quality factor and by the shift of the resonance frequency to lower values.

5) The mathematical expressions for the quality factor can be further used for an analysis of the resonator sensitivity to geometrical imperfections (small imperfections of the radius a and the length c).

5. Conclusions

The study conducted in this paper demonstrates the importance of maintaining a “clean” cavity inside the electromagnetic resonator, in order to obtain a high quality factor. The results are presented as mathematical expressions (not analytical expressions) since they require numerical quadratures for the evaluation of Q .

The expressions established for Q will be used in a further optimization and sensitivity study.

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UNELE ASPECTE REFERITOARE LA FACTORUL DE CALITATE AL CAVITĂȚILOR REZONANTE CILINDRICE

(Rezumat)

Lucrarea prezintă modul de calcul exact al factorului de calitate, Q , pentru rezonatorul cilindric cu pereți metalici, luând în calcul și posibilitatea ca dielectricul să aibă pierderi. Expresiile matematice obținute presupun realizarea unei integrări numerice pentru evaluarea lui Q . Se face o simulare pentru a studia influența permitivității și a tangentei unghiului de pierderi asupra lui Q în modurile TM și TE, considerând dielectrics cu pierderi precum apa distilată, gheața și depunerile de sol nisipos.